UNIVERSITY OF BRISTOL
FACULTY OF ENGINEERING

Second Year Examination for the Degrees of
Bachelor and Master of Engineering

MAY/JUNE 2010 3 Hours

SPECIMEN PAPER

ENGINEERING MATHEMATICS 2

EMAT20200

This paper contains *ten* questions; *five* in Section A and *five* in Section B
Answer *all* questions in Section A, and choose *four* from Section B

Answer *nine* questions in *three* hours

Questions in Section A are worth *eight* marks each
Questions in Section B are worth *twenty* marks each
The maximum mark for this paper is *120 marks*

The Following data sheets are attached:

Facts and Formulae
Normal Probability Tables
Chi-Squared Probability Density Function
T-Probability Density Function
Definitions and Formulae in Statistics
Fourier Series
Laplace Transforms

Calculators must have the Faculty of Engineering's Seal of Approval

SECTION A

Q1 (a) On an uncrowded motorway, 1.2 vehicles per second on average are observed to be passing under a bridge.
What is the probability of observing

(i) no vehicles over two seconds
(ii) two or more vehicles over three seconds

(4 marks)

(b) The number of vehicles is observed on film every second over a one minute period. Determine the probability that more than 80 vehicles will pass under the bridge.

(4 marks)

Note that a random variable X follows a Poisson distribution of mean μ and satisfies

$$P(X = x) = \frac{\mu^x e^{-\mu}}{x!}; \quad 0! = 1.$$

Q2 A furnace being calibrated to determine how long it takes for it to warm up from 0°C to a specified temperature yields the following table of results

Time in minutes (x)	10	20	30	40	50	60
Temperature (°C)	5	42	67	82	100	126

Assuming the data to be linear, use linear regression to find the temperature after a given time interval (given to 3dp). Assume that the equation of the line of regression of y on x, i.e. $y = \hat{\alpha} + \hat{\beta} x$, can be determined using

$$\hat{\alpha} = \frac{\sum y_i \sum x_i^2 - \sum x_i y_i \sum x_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$\hat{\beta} = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \quad ; \quad i = 1(1)n$$

(8 marks)

Q3 (a) Show that the vector field

$$\boldsymbol{v} = (2y \cos x \sin x - \sin^2 z)\ \boldsymbol{i} + \sin^2 x\ \boldsymbol{j} - 2x \cos z \sin z\ \boldsymbol{k}$$

is conservative, that is, show that curl $\boldsymbol{v} = 0$.

(2marks)

(b) Find a scalar potential ϕ such that $\boldsymbol{v} = \text{grad}\ \phi$.

(4 marks)

(c) Using (b) calculate the work integral $\int \boldsymbol{v}.d\boldsymbol{r}$ along the straight line from the point (0, 0, 0) to the point (π, π, π).

(2 marks)

Q4 (a) Prove that there is a solution to equation

$$\sin x - \cos 2x = 0 \qquad \text{on } [0, \frac{\pi}{2}]$$

(3 marks)

(b) Use the interval bisection method to estimate the solution to within

$$0.2 \simeq \frac{\pi}{16}$$

(3marks)

(c) Refine your estimate using one iteration of Newton's method. Why do you expect this to be a better answer than one more bisection would be?

(2marks)

Q5 Use D'Alembert's method to find the solution to the wave equation

$$u_{tt} = c^2 u_{xx}$$

on the interval $-\infty < x < \infty$ subject to the initial conditions

$$u(x,0) = e^{-x^2} \cos(x), \quad u_t(x,0) = 0$$

sketch the solution *u(x,t)* as a graph of *u* against *x* for $t = 0$, $t = 1$ and for large time.

(8 marks)

turn over/…

SECTION B

Q6 Consider the volume V given by $x \geq 0,\ y \geq 0,\ x + y \leq r$ and $0 \leq z \leq r$. To determine the centre of gravity of V, assuming that the density $\vartheta(x, y, z) \equiv 1$, proceed as follows:

(a) Sketch V.

(4 marks)

(b) Determine the mass M of V.

(2 marks)

(c) Calculate the components $\bar{x}, \bar{y}$ and $\bar{z}$ of the centre of gravity $(\bar{x}, \bar{y}, \bar{z})$ of V.

(14 marks)

Q7 Consider the solid cylinder V given by $x^2 + y^2 \leq 1$ and $-1 \leq z \leq 1$. For the vector field $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ verify the Grauss Divergence Theorem as follows:

(a) Sketch the solid V and its bounding surface S.

(4 marks)

(b) Calculate the volume integral $\iiint_V div\boldsymbol{F}dV$.

(3 marks)

(c) Calculate the surface integral $\iint_S \boldsymbol{F.n}dA$.

(13 marks)

Q8 (a) When a standard lubricant is applied to a wheel bearing, the length of time for which the wheel will continue to spin is approximately Normally distributed with a mean of 200 seconds. Another lubricant, believed to be better, is tried out and five test spins take the following times (in seconds)

216, 211, 208, 226, 194

Test at the 5% level whether the new lubricant has significantly increased the spinning times.

(8 marks)

(b) The number of fatal road accidents in a major city over a one year period is shown in the table below, according to the time of occurrence.

Time	Number
0000-0400	22
0400-0800	10
0800-1200	13
1200-1600	14
1600-2000	24
2000-0000	31

turn over/Qu. continues…

(cont.)

(i) Test the hypothesis that the number of such accidents is uniformly distributed in time at the 1% level.

(6 marks)

(ii) The number of road miles covered by all vehicles for the six time intervals are in the ratio

$$2 : 1 : 4 : 3 : 5 : 3$$

Test the hypothesis that the number of accidents is distributed in these ratios, again using a 1% level of significance.

(6 marks)

Q9 Consider the initial value problem

$$\frac{dx}{dt} = \frac{e^{-\frac{t^2}{2}}}{2x}, \quad x(0) = 1$$

(a) Use Euler's method with 3 steps to estimate $x(1)$.

(2marks)

(b) Show that $x(t) = \left\{ \int_0^t e^{-\frac{s^2}{2}} ds + 1 \right\}^{\frac{1}{2}}$ is a solution to the I.V.P in (a).

(3 marks)

(c) Use the trapezium rule to estimate $x(1)$, with two panels.

(3 marks)

(d) Use Simpson's rule with 2 panels to estimate $x(1)$. Which estimate do you expect to be best, and why?

(3 marks)

(e) Consider the following fixed point schemes

(A) $x = g_1(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}}$

(B) $x = g_2(x) = \left(\frac{10}{4 + x} \right)^{\frac{1}{2}}$

(i) Check that the fixed points $x_1^* = g_1(x_1^*)$ and $x_2^* = g_2(x^*)$ are the same.

(3 marks)

(ii) Complete 3 iterations of each scheme, starting at $x_0 = 1.5$

(2 marks)

(iii) Use the theory of convergence of fixed point schemes to predict which will converge faster and by how much, on values in [1, 1.5].

(4 marks)

turn over/…

Q10 (a) Use the method of Laplace transforms to find the solution to the following ordinary differential equations

$$\frac{dx}{dt} + 3x(t) = e^{-3t}, \quad x(0) = 1$$

(5 marks)

(b) The Fourier transform $F(\omega)$ of a function $f(t)$ is defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-jwt}\, dt$$

Let $\Im$ represent the action of taking the Fourier transform, that is, $\Im[f(t)] = F(\omega)$.

(i) Suppose that $F(\omega)$ is the Fourier transform of $f(t)$, and that $f(t)$ decays rapidly to zero as $t \to \pm\infty$, then show from first principles that

$$\Im\left[\frac{df}{dt}\right] = j\omega F(\omega)$$

(3 marks)

(ii) Calculate the Fourier transform $F(\omega)$ for the function

$$f(t) = e^{-|t|}$$

(where $|t|$ represents the absolute value of t), expressing the answer in the simplest possible form.

(5 marks)

(iii) Consider the following ordinary differential equation with input *u(t)* and output *y(t)*

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 7y(t) = u(t)$$

Let $\Im[y(t)] = Y(\omega), \Im[u(t)] = U(\omega)$. Take the Fourier transform of the equation, and hence find the frequency transfer function $G(\omega)$ such that $Y(\omega) = G(\omega)U(\omega)$.

(4 marks)

(iv) Hence find the frequency response function $Y(\omega)$ in the case where *u(t)* is the function *f(t)* given in part (ii). Find also the magnitude spectrum $|Y(\omega)|$.

(3 marks)