

# EMaII Vector Calculus

## Example Sheet 8: Stokes' Theorem and polar co-ordinates

2002/3

1 Verify Stokes' Theorem for

$$\mathbf{v} = (y - z + 2)\mathbf{i} + (yz + 4)\mathbf{j} - xz\mathbf{k}$$

where  $S$  is the surface of the cube bounded by  $x = 0$ ,  $y = 0$ ,  $z = 0$ ,  $x = 2$ ,  $y = 2$ ,  $z = 2$  with the face in the  $(x, y)$ -plane (i.e.  $z = 0$ ) missing

2 Use Stokes' Theorem to evaluate the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where  $C$  is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and  $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

3 Use volume integration to calculate the volume in the first octant (region of positive  $x$ ,  $y$ ,  $z$ ) enclosed by the surface  $x^2 + y^2 + z^2 = a^2$ . Show that the position of the centre of gravity is given by  $(\bar{x}, \bar{y}, \bar{z}) = \frac{3a}{8}(1, 1, 1)$ .

4 Use cylindrical polar co-ordinates to evaluate

$$\int \int \int_V \sqrt{x^2 + y^2 + z^2} dx dy dz$$

where  $V$  is the region bounded by the plane  $z = 3$  and the cone  $z = \sqrt{x^2 + y^2}$ .

5(a) Use the ellipsoidal polar co-ordinate system defined by

$$x = ar \sin \theta \cos \phi, \quad y = br \sin \theta \sin \phi, \quad z = cr \cos \theta,$$

in order to evaluate

$$\int \int \int_V (x^2 + y^2 + z^2) dx dy dz$$

where  $V$  is the interior of the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

(Hint, first calculate the volume factor  $|J|$ ).

(b) Is the co-ordinate system defined in part (a) curvilinear?

6 Use the definitions of grad, div and curl in polar co-ordinates given in the notes to evaluate

(a)  $\text{grad } \phi$ , where  $\phi = \frac{1}{|\mathbf{r}|}$

(b)  $\text{curl } \mathbf{v}$ , where  $\mathbf{v} = r^2 \cos \theta \mathbf{e}_r + \frac{(\sin \theta \mathbf{e}_\theta + \mathbf{e}_\phi)}{r \sin \theta}$

(c)  $\text{div } \mathbf{v}$ , for the same vector field as part (b)

Numerical answers to follow in the lectures.