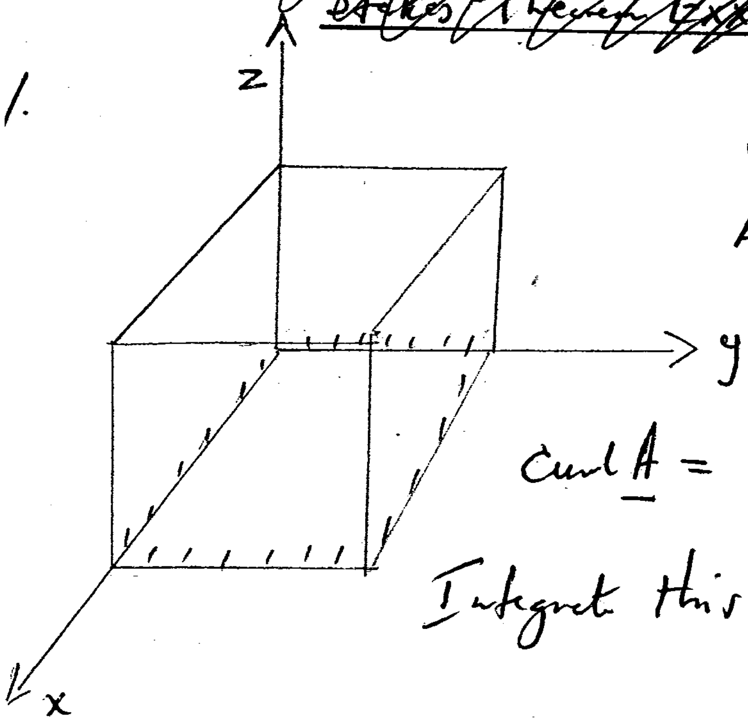
~~10 Stokes Theorem Exs~~

Vector Calculus (6) Ex Sheet 8 Solns

1.



Cube with xy plane face missing.

$A = (y - z + 2)\underline{i} + (yz + 4)\underline{j} - xz\underline{k}$

$\text{curl}\,\underline{A} = (\,y,\ -1 + z,\ 1\,)$

Integrate this function over the surfaces.

a. $x = 0,\ \underline{\hat{n}} = -\underline{i}$ $\quad \text{curl}\,\underline{A}\cdot\underline{\hat{n}} = -y$

$$-\int_0^2\int_0^2 y\,dz\,dy = -\left[\frac{y^2}{2}\right]_0^2\left[z\right]_0^2 = -4$$

b. $y = 0,\ \underline{\hat{n}} = -\underline{j}$, $\quad \text{curl}\,\underline{A}\cdot\underline{\hat{n}} = 1 - z$

$$-\int_0^2 (z-1)\,dz\,dx = -\left(\frac{z^2}{2} - z\right]_0^2\left[x\right]_0^2 = 0.$$

c. $x = 2,\ \underline{\hat{n}} = \underline{i}$ $\quad \text{Curl}\,\underline{A}\cdot\underline{\hat{n}} = y$

$\iint \underline{A}\cdot\underline{\hat{n}}\,dz\,dy = +4.$ — cancels with a.

d. $y = 2,\ \underline{\hat{n}} = \underline{j}$, $\quad \text{curl}\,\underline{A}\cdot\underline{\hat{n}} = (z-1)$

$\iint \text{curl}\,\underline{A}\cdot\underline{\hat{n}}\,dz\,dx = 0.$

NB. Refers to curl $\underline{A}$, not $\underline{A}$

e. $z = 2,\ \underline{\hat{n}} = \underline{k}.$ $\quad \text{Curl}\,\underline{A}\cdot\underline{\hat{n}} = 1.$ $\quad \therefore \iint \text{Curl}\,\underline{A}\cdot\underline{\hat{n}}\,dx\,dy = \text{area} = 4.$

Adding up gives — $-4 + 4 + 4 = 4.$

1. ctd.

In the $z=0$ plane -

$\int \underline{A} \cdot d\underline{r}$ becomes -

$$\underset{\substack{(y=0)\\ z=0}}{\int_0^2 A_x\,dx} + \underset{\substack{(x=2)\\ z=0}}{\int_0^2 A_y\,dy} + \underset{\substack{(y=2\\ z=0)}}{\int_2^0 A_x\,dx} + \underset{\substack{(x=0\\ z=0)}}{\int_2^0 A_y\,dy}$$

$$= \int_0^2 2dx + \int_0^2 4dy + \int_2^0 4\,dx + \int_2^0 4\,dy$$

$$= -2\int_0^2 dx = -4.$$

(In this sense the integral has be taken clockwise whereas the usual sense is anticlockwise. Taking the integral anticlockwise would change the sign making it equal to the value of the surface integral.)

2. $\oint_C \underline{B}.d\underline{r}$ where C is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Using stokes theorem, we can find

$\iint_S \text{curl}\,\underline{B}.d\underline{S}$ $d\underline{S}$ //'l $\underline{k}$, normal to the plane.

(open surface)

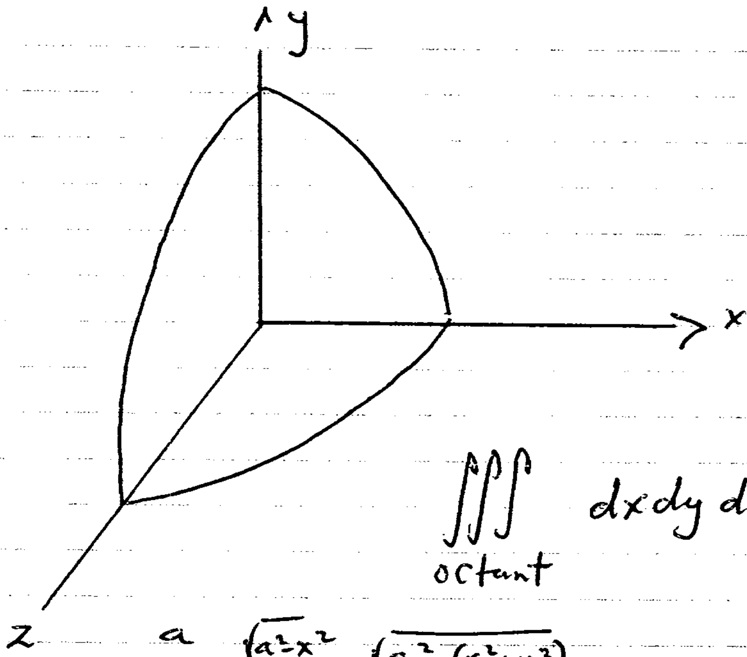
If $\underline{B}= -y\underline{i}+x\underline{j}$, (in x-y plane)

$$\text{curl}\,\underline{B} = \left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}, \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}, \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)$$

$$= (0,0,1+1) = (0,0,2). \quad (2\underline{k}).$$

$\therefore \iint_S 2\underline{k}.\underline{k}\,dx\,dy = 2\times$ ellipse area $= 2\pi ab.$

3.



$$\iiint_{\text{octant}} dx\,dy\,dz$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-(x^2+y^2)}} dz\,dy\,dx.$$ Taking z first – but any could be taken.

In spherical polars we have

$$\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta \; d\phi \; d\theta \; dr$$ $(0 \le \phi \le \pi/2)$ $(0 \le \theta \le \pi/2)$.

$$= \left[\frac{r^3}{3}\right]_0^a \left[-\cos\theta\right]_0^{\pi/2} \left[\phi\right]_0^{\pi/2}$$

$$= \frac{a^3}{3} \cdot 1 \cdot \frac{\pi}{2} = \frac{\pi a^3}{6}$$ ∴ Total spherical volume = $8 \cdot \frac{\pi a^3}{6} = \frac{4}{3}\pi a^3$.

Additional –

c.g. of octant. $(\bar{x}=\bar{y}=\bar{z})$.

$$\bar{x} = \frac{\rho \iiint_{vol} x \, dz\, dy\, dx}{\rho \iiint_{vol} = \text{mass}}$$

$$\therefore \frac{\pi a^3}{6}\bar{x} = \int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r \sin\theta \cos\phi \, . \, r^2 \sin\theta \; d\phi\, d\theta\, dr.$$

$$= \left[\frac{r^4}{4}\right]_0^a \left[\frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} \left[\sin\phi\right]_0^{\pi/2}$$

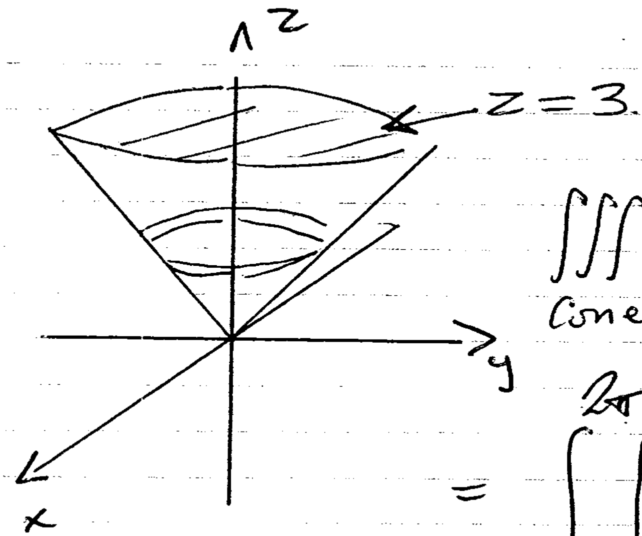
$$= \frac{\pi a^4}{16}, \quad \therefore \bar{x} = \frac{3a}{8} = \bar{y} = \bar{z}.$$

As C.G. $(\bar{x},\bar{y},\bar{z})$ is at $\frac{3a}{8}(1,1,1)$

it follows by symmetry that the other octants have C.G. at. $\frac{3a}{8} \times \quad (-1,1,1),\ (1,-1,1),\ (-1,-1,1)$.

and that the C.G of a complete solid hemisphere is at $\frac{3a}{8}(0,0,1)$ or $\frac{3}{8}$ radius above the base on ⊥r radius to the centre.

4.



$$\iiint_{\text{Cone}} \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$$

($z=\rho$, polars)

$$= \int_0^{2\pi}\int_0^{3}\int_0^{z} \sqrt{\rho^2+z^2}\cdot \rho \cdot d\rho\, dz\, d\theta$$

As the cone has equation $z=\rho$ (polars), $0 \le \rho \le z$.
also the semi-angle $= 45^\circ$ $(\pi/4)$.

$$\int_0^{z} (\rho^2+z^2)^{1/2} \rho\, d\rho = \frac{1}{2}\cdot\frac{2}{3}(\rho^2+z^2)^{3/2}$$

$$= \frac{1}{3}\left\{(2z^2)^{3/2} - (z^2)^{3/2}\right\}$$

$$= \frac{z^3}{3}(2\sqrt{2}-1).$$

$$\int_0^{2\pi} d\theta \int_0^{3} \frac{(2\sqrt{2}-1)}{3} \cdot z^3\, dz = 2\pi \cdot \frac{3^4}{4}\cdot\frac{(2\sqrt{2}-1)}{3}$$

$$= \frac{27\pi(2\sqrt{2}-1)}{2}$$

5) $x = ar\sin\theta\cos\phi$
$y = br\sin\theta\sin\phi$ = elliptical polars
$z = cr\cos\theta$ (note NOT orthogonal)

$$J = \begin{vmatrix} a\sin\theta\cos\phi & ar\cos\theta\sin\phi & -ar\sin\theta\sin\phi \\ b\sin\theta\sin\phi & -br\cos\theta\sin\phi & -br\sin\theta\cos\phi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix}$$

$= abcr^2\sin\theta$ (after some algebra)

$$\therefore \iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^4 abc\sin\theta\left(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta\right) dr\,d\theta\,d\phi$$

$$= abc\int_0^1 r^4\,dr\left[\int_0^{2\pi}\int_0^{\pi}\sin\theta\left(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta\right)d\theta\,d\phi\right.$$

$= \dfrac{4\pi}{15}abc\,(a^2+b^2+c^2)$ after some algebra.

6) a) $\varphi = \frac{1}{|\underline{r}|} = \frac{1}{r}$ in spherical polar co-ords

$\text{grad } \varphi = \underline{e}_r \frac{\partial}{\partial r}\left(\frac{1}{r}\right)$ (in spherical polars)

$= -\frac{1}{r^2}\underline{e}_r$

$= -\frac{\underline{r}}{|\underline{r}|^3}$

b)

$\underline{v} = r^2\cos\theta\, \underline{e}_r + \underline{e}_\theta \sin\theta + \frac{\underline{e}_\varphi}{r\sin\theta}$

scale factors

$h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$

$$\therefore \text{curl } \underline{v} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & r\sin\theta\,\underline{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ r^2\cos\theta & 1 & 1 \end{vmatrix}$$

multiplying components of $\underline{v}$ by scale factors

$= r\sin\theta\, \underline{e}_\varphi$

c) $\text{div } \underline{v} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\, r^2\cos\theta\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta \cdot \frac{1}{r}\right) + \frac{\partial}{\partial \varphi}\left(r \cdot \frac{1}{r\sin\theta}\right)\right]$

$= 4r\cos\theta + \frac{1}{r^2\tan\theta}$