Summary

1. **Introduction** The difference between scalar functions, vector functions, scalar fields and vector fields. What they represent graphically and physically. Differentiation and integration of vector functions.

2. **Differentiation of Scalars**

   - grad ϕ ($\boldsymbol{\nabla}\phi$); its interpretation in terms of slope on a contour plot; its use to define directional derivative,
   - Use of grad to find maxima, minima and saddles, including use of Hessian matrix to determine which; the Lagrange multiplier method for constained optimisation

3. **Differentiation of Vectors**

   - div $\boldsymbol{v}$ ($\boldsymbol{\nabla} \cdot \boldsymbol{v}$); its physical interpretation; divergence-free = incompressible fluid, solenoidal force fields; proof $\text{div}\text{curl}\boldsymbol{v} \equiv 0$.
   - curl $\boldsymbol{v}$ ($\boldsymbol{\nabla} \times \boldsymbol{v}$); its physical interpretation; curl-free = irrotational fluid, conservative force; curl $\boldsymbol{v} = 0 \Leftrightarrow \boldsymbol{v} = \boldsymbol{\nabla}\phi$ (proof in $\Leftarrow$ direction only); construction of potential ϕ if curl $\boldsymbol{v} = 0$.
   - derivation of simple differentiation rules and formulae using Cartesian co-ordinates; definition of the Laplacian $\nabla^2\phi$ in Cartesian co-ordinates, and that $\nabla^2 = \text{div grad}$.

4. **Path integration**

   - The intrinsic parametrisation of curves $\boldsymbol{r}(t)$; concept and simple examples; arclength $S := \int_a^b (dr/dt)dt$.
   - **Scalar path integral** $\int_C f ds = \int_C f|(dr/dt)|dt$. Evaluate by parametrising C with $r(t)$, evaluating f as a function of t and calculating limits on t.
   - **Work integral** $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \boldsymbol{F} \cdot (d\boldsymbol{r}/dt)dt$; evaluate as in scalar case; application to work done; independence of path

property conservative ⇒ curl $\boldsymbol{F}$ ⇔ $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$; proof of ⇒; closed curves $\oint \boldsymbol{F} \cdot d\boldsymbol{r}$ ($= 0$ for conservative $\boldsymbol{F}$).

5. **Double and triple integrals**

   - **Double integrals** $\int\int_R f\,dxdy$; changing the order ⇒ changing the limits for non-rectangular domains; seperable in rectangular domains if $f = f(x)g(y)$. Application to area, mass, centres of gravity and moments of inertia. Changing co-ordinates to plane polars ⇒ $dxdy = rdrd\theta$.
   - **Triple integral** $\int\int\int_V f\,dxdydz$; work out limits carefully (draw a picture!) inner limits function of outer two variables, middle limit function of outer variable, outer limits constant. Similar properties to double integrals; applications to 3D area, mass, centres of gravity and moments of inertia. Use of the Jacobian as a volume element; cylindrical polar co-ordinates $dxdydz = rdrd\theta dz$; spherical polars $dxdydz = r^2 \sin\theta drd\theta d\phi$

6. **Surface integrals**

   - Parametrisation of surfaces $\boldsymbol{r}(u, v)$; orientation of surfaces (outward and inward); normal vector $\hat{n}$ in direction $\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$.
   - **Flux integral** $\int_S \boldsymbol{v} \cdot d\boldsymbol{A}$ and **scalar surface integral** $\int_S \boldsymbol{v} |dA|$; evaluation by calculating $d\boldsymbol{A} = \hat{\boldsymbol{n}} dA = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} dudv$ and calculating limits on u and v.

7. **The fundamental theorems**

   - Gauss's **Divergence Theorem** for a *closed* surface S bounding a volume V; statement; application to evaluation of surface integrals; application to the physical meaning of div.
   - **Stokes' Theorem** for a curve C bounding an *open* surface S; statement; Green's Theorem as a special case; application to evaluation of line integrals; application to the physical meaning of curl; proof of independence of path result.