

Some corrections and completions ...

Worked example 3.1 (last part) *Prove*

$$L[tf(t)] = -\frac{dF}{ds}$$

Use the combined results to find the Laplace transform of $t^2 \sin(\omega t)$.

Consider the right-hand side

$$\begin{aligned} -\frac{dF}{ds} &= -\frac{d}{ds} \int_{t=0}^{\infty} f(t)e^{-st} dt \\ &= -\int_{t=0}^{\infty} \frac{d}{ds}(f(t)e^{-st}) dt \\ &= -\int_{t=0}^{\infty} f(t) \frac{d}{ds}(e^{-st}) dt \\ &= -\int_{t=0}^{\infty} f(t)(-te^{-st}) dt \\ &= \int_{t=0}^{\infty} tf(t)e^{-st} dt \\ &= L[tf(t)] \quad \text{Q.E.D.} \end{aligned}$$

Hence $L[t^2 f(t)] = +\frac{d^2 F(s)}{ds^2}$. So we can do the last part of the questions

$$\begin{aligned} L[t^2 \sin(\omega t)] &= \frac{d^2}{ds^2} L[\sin(\omega t)] \\ &= \frac{d^2}{ds^2} \left[\frac{\omega}{s^2 + \omega^2} \right] \\ &= \frac{d}{ds} \left[\frac{-2s\omega}{(s^2 + \omega^2)^2} \right] \\ &= \frac{8\omega s^2}{(s^2 + \omega^2)^3} - \frac{2\omega}{(s^2 + \omega^2)^2} \end{aligned}$$

Worked example 3.5 Use Laplace transforms to solve the show that the solution of the system of ODEs

$$\begin{aligned} \frac{dx}{dt} + 2\frac{dy}{dt} + x - y &= 5 \sin t \\ 2\frac{dx}{dt} + 3\frac{dy}{dt} + x - y &= e^t \end{aligned}$$

subject to the initial conditions $x(0) = 2$, $y(0) = 1$ is

$$\begin{aligned} x(t) &= 5 \sin t + 5 \cos t - e^t - (1/2)e^{2t} - 3/2 \\ y(t) &= 2e^t - 5 \sin t + ((1/2)e^{2t} - (3/2)) \end{aligned}$$

Note that the algebra for this example is a brute and is probably much harder than you might expect in an exam!

Consider the first equation. Taking transforms of both sides, we get

$$(sX(s) - x(0)) + 2(sY(s) - y(0)) + X(s) - Y(s) = \frac{5}{1 + s^2}$$

Similarly, for the second equation

$$2(sX(s) - x(0)) + 3(sY(s) - y(0)) + X(s) - Y(s) = \frac{1}{s - 1}$$

Rearranging and using the initial conditions we get

$$(s + 1)X(s) + (2s - 1)Y(s) = 4 + \frac{5}{1 + s^2} \quad (1)$$

$$(2s + 1)X(s) + (3s - 1)Y(s) = 7 + \frac{1}{s - 1} \quad (2)$$

Multiplying (1) by $(2s + 1)$ and the (2) by $(s + 1)$ we get

$$(2s + 1)(s + 1)X(s) + (4s^2 - 1)Y(s) = 4(2s + 1) + \frac{10s + 5}{1 + s^2}$$

$$(2s + 1)(s + 1)X(s) + (3s - 1)(s + 1)Y(s) = 7(s + 1) + \frac{s + 1}{s - 1}.$$

Subtracting these two equations, we get

$$(4s^2 - 1 - 3s^2 - 2s + 1)Y(s) = 4(2s + 1) - 7(s + 1) + \frac{10s + 5}{1 + s^2} - \frac{s + 1}{s - 1},$$

which simplifies to

$$s(s - 2)Y(s) = s - 3 + \frac{10s + 5}{1 + s^2} - \frac{s + 1}{s - 1}.$$

Hence

$$Y(s) = \frac{s - 3}{s(s - 2)} + \frac{10s + 5}{s(s - 2)(1 + s^2)} - \frac{s + 1}{s(s - 1)(s - 2)}.$$

To take the inverse transform we need to express $Y(s)$ in terms of partial fractions. We obtain

$$\begin{aligned} -\frac{s + 1}{s(s - 1)(s - 2)} &= -\frac{3}{2s} + \frac{1}{2(s - 1)} \\ \frac{10s + 5}{s(s - 2)(1 + s^2)} &= \frac{5}{2(s - 2)} - \frac{5}{2(1 + s^2)} - \frac{5}{2s} \\ \frac{s - 3}{s(s - 2)} &= \frac{3}{2s} - \frac{1}{2(s - 2)} \end{aligned}$$

Hence

$$Y(s) = \frac{2}{s - 1} + \frac{1}{s(s - 2)} - \frac{3}{2s} - \frac{5}{1 + s^2}$$

Similarly, we can obtain $X(s)$ by taking $(3s - 1)$ times (1) and subtracting $(2s - 1)$ times (2). In so doing, we obtain

$$-s(s - 2)X(s) = 4(3s - 1) + \frac{5(3s - 1)}{1 + s^2} - 7(2s - 1) - \frac{(2s - 1)}{s - 1}.$$

This can be rearranged to give

$$X(s) = \frac{2s - 3}{s(s - 2)} + \frac{2s - 1}{s(s - 1)(s - 2)} - \frac{15s - 5}{s(s - 2)(1 + s^2)}.$$

After a lot of work we can express the right-hand side using partial fractions:

$$X(s) = \frac{5s + 5}{1 + s^2} - \frac{1}{2(s - 2)} - \frac{3}{2s} - \frac{1}{s - 1}.$$

So, finally we need to convert back using the table of inverse Laplace transforms. We get

$$\begin{aligned} x(t) &= 5 \sin(t) + 5 \cos(t) - \frac{3}{2} - \frac{1}{2}e^{-2t} - e^{-t}, \\ y(t) &= 2e^t + \frac{1}{2}e^{2t} - \frac{3}{2} - 5 \sin(t). \end{aligned}$$

Worked example 3.8 *If the spring modulus is λ , its natural length is a and the damping coefficient is δ then the equation of motion of a spring-mass damper system is:*

$$m \frac{d^2x}{dt^2} = u - \lambda(x - a) - \delta \frac{dx}{dt}.$$

Show that the transfer function $G(s)$ of this linear system is

$$\frac{1}{(ms^2 + \delta s + \lambda)}.$$

We can rewrite the equation in the form

$$m \frac{d^2x}{dt^2} + \delta \frac{dx}{dt} + \lambda x = u + \lambda a$$

where we think of a new input as being $u + \lambda a$. Alternatively to create a linear system without a constant term added to the input (so that the output will be centred on zero) we can define a new variable $y = x - a$. Then note that $\frac{dy}{dt} = \frac{dx}{dt}$ and $\frac{d^2y}{dt^2} = \frac{d^2x}{dt^2}$. Therefore we get the new system

$$m \frac{d^2y}{dt^2} + \delta \frac{dy}{dt} + \lambda y = u$$

The transfer function is then obtained by taking Laplace transforms and assuming that at time $t = 0$ the system is in equilibrium, that is $x = a$, $\frac{dx}{dt} = 0$. That is $y(0) = 0$, $\frac{dy}{dt}(0) = 0$. We get

$$m(s^2Y(s) - sy(0) - y'(0)) + \delta(sY(s) - y(0)) + \lambda Y(s) = U(s)$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $U(s)$ the Laplace transform of $u(t)$.

Hence we get

$$ms^2Y(s) + \delta sY(s) + \lambda Y(s) = U(s),$$

which gives

$$(ms^2 + \delta s + \lambda)Y(s) = U(s).$$

Hence,

$$Y(s) = \frac{1}{ms^2 + \delta s + \lambda}U(s)$$

where

$$G(s) := \frac{1}{ms^2 + \delta s + \lambda}$$

is the transfer function.