6. Characteristics and d'Alembert's method

d'Alembert's method for the wave equation on an infinite domain. Examples. General solution. d'Alembert's method for semi-infinite domain.

6.1 General solution of the wave equation

The separation of variables method is one way of finding solutions of the wave equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

It is well suited to the case where we have boundary conditions. Then the 'wavelength' is determined by the boundary conditions. For example, what is the note played by the guitar string, the organ pipe or a percussive instrument? The solution is said to be a *standing wave.*

This is not a viable method if the domain over which we solve the PDE is infinite.Of course, nothing is truly infinite (except in mathematics and religion). Really, what we mean by an infinite domain is a long domain in which the boundaries are far away and cannot influence the wave length. For example, what is the shape of ripples if we drop a stone into a pond? How to waves propagate along a long cable — in a cable stayed bridge, bacterial flagellum or a whip ? The kind of solution we are looking for is a *travelling wave.*

In order to find travelling wave solutions we note the following result.

Theorem *A general solution of the wave equation can be expressed in the form*

$$u(x,t) = f(x-ct) + g(x+ct)$$

for arbitrary functions f and g.

Proof just differentiate twice with respect to t and x.

Let $u(x,t) = f(x-ct) + g(x+ct)$, then

$$\begin{aligned} u_x &= f'(x-ct) + g'(x+ct) \\ \Rightarrow u_{xx} &= f''(x-ct) + g''(x+ct) \end{aligned}$$

and

$$\begin{aligned} u_t &= -cf'(x-ct) + cg'(x+ct) \\ \Rightarrow u_{tt} &= c^2 f''(x-ct) + c^2 g''(x+ct)\,. \end{aligned}$$

Hence $u_{tt} = c^2 u_{xx}$ $(= f''(x-ct) + g''(x+ct))$. $\Box$.

Remarks

1. This is known as **d'Alembert's solution** to the wave equation.
2. Note what functions $f(x-ct)$ and $g(x+ct)$ look like. Functions of the form $f(x-ct)$ represent waves travelling *to the right* and $g(x+ct)$ waves travelling *to the left.*

   Worked example 6.1 *Sketch the function* $\sin(x-ct)$ *for* $t=0$, $ct=\pi/4$, $ct=\pi/2$, $ct=3\pi/4$ *and* $ct=\pi$. *Show that this represents a wave that travels to the right.*

3. The form that f and g take is determined by the boundary conditions. We make a distinction between waves on infinite domains (dropping a stone in a pond) and on semi-infinite domains (wave propagation along a whip).

6.2 Method for an infinite domain

Example: *Consider the wave equation on an infinite domain*

$$u_{tt} = c^2 u_{xx}, \qquad -\infty \le x \le \infty, \quad t \ge 0,$$

subject to the plucked *initial conditions*

$$u_t(x,0) = 0, \qquad u(x,0) = F(x) := \begin{cases} 1+x & -1 \le x < 0, \\ 1-x & 0 \le x < 1 \\ 0 & \textit{otherwise} \end{cases}.$$

Note there are no boundary conditions, just initial conditions. The boundary conditions at $x = \pm\infty$ are implicit (the solution should be finite as $x \to \pm\infty$).

Step 1: state the d'Alembert solution.

$$u(x,t) = f(x - ct) + g(x + ct)$$

Step 2: use the initial conditions. First we take the zero-derivative condition

$$0 = u_t(x,0) = \left[-cf'(x - ct) + cg'(x + ct)\right]_{t=0},$$

which implies that

$$-f'(x) + g'(x) = 0.$$

We can integrate this expression with respect to x and we get

$$g(x) - f(x) = K, \tag{6.1}$$

for some constant K.

Now, solving the initial condition on displacement we get

$$F(x) = u(x,0) = f(x) + g(x),$$

where $F(x)$ was the known function given by the initial condition. Hence

$$f(x) + g(x) = F(x). \tag{6.2}$$

Step 3: solve the simultaneous equations. Equations (6.1) and (6.2) represent two simultaneous equations for the unknown functions f and g. Substituting $g(x) = K + f(x)$ from (6.1) into (6.2), we get

$$2f(x) + K = F(x)$$

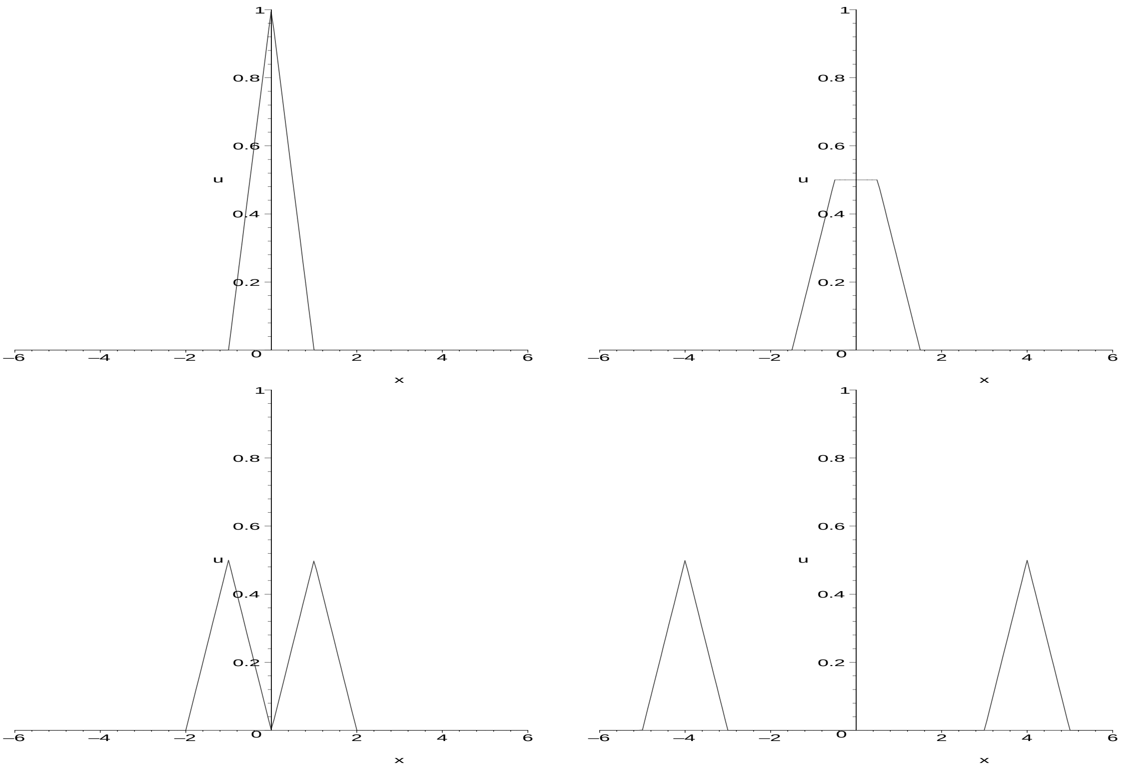
Therefore

$$f(x) = \frac{1}{2}F(x) - \frac{K}{2} \quad \text{and} \quad g(x) = \frac{1}{2}F(x) + \frac{K}{2}.$$

Step 4: recombine to get general solution. We have

$$\begin{aligned} u(x,t) &= f(x-ct)+g(x+ct) \\ &= \frac{1}{2}F(x-ct)-\frac{K}{2}+\frac{1}{2}F(x+ct)+\frac{K}{2} \\ &= \frac{1}{2}F(x-ct)+\frac{1}{2}F(x+ct) \end{aligned}$$

Step 5: plot the solution profile. Plots $u(x,t)$ for ct = 0, 0.5, 1, 4.



We can generalise this solution method to solve the wave equation on an infinite domain subject to any initial displacement $F(x)$ and initial velocity $G(x)$.

Worked example 6.2 *Show that the general solution to the wave equation*

$$u_{tt} = c^2 u_{xx}, \qquad -\infty \leq x \leq \infty, \quad t \geq 0,$$

subject to the initial conditions

$$u(x,0) = F(x), \qquad u_t(x,0) = G(x),$$

is

$$u(x,t) = \frac{1}{2}[F(x-ct) + F(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$

Find the specific solution in the case of an impulsive initial velocity $F(x) = 0$*,* $G(x) = e^{-|t|}$*.*

6.3 Method for a semi-infinite domain

On a semi-infinite domain, we often have a slightly more involved process.

Example: Consider the wave equation on a semi-infinite domain

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le \infty, \quad t \ge 0,$$

subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0$$

and boundary condition

$$u(0,t) = \sin(\omega t),$$

which corresponds to a long string having one subjected to a time-dependent excitation.

Note there is now only one boundary condition (in addition to the two initial conditions). Ordinarily for the wave equation we would expect 2 boundary conditions. The other boundary condition is an implicit one at $x = +\infty$ that the solution should be finite as $x \to \infty$.

The solution to begin with proceeds as before, for the fully infinite domain, but with care to only allow x to be positive.

Step 1: state the d'Alembert solution.

$$u(x,t) = f(x-ct) + g(x+ct)$$

Step 2: use the initial conditions. First we take the zero-derivative condition

$$\begin{aligned} 0 &= u_t(x,0) = \left[-cf'(x-ct) + cg'(x+ct)\right]_{t=0}, \\ \Rightarrow 0 &= -f'(x) + g'(x) \end{aligned}$$

$$\Rightarrow K = g(x) - f(x), \tag{6.3}$$

for some constant K.

Setting the initial displacement to zero, we get

$$0 = u(x,0) = f(x) + g(x). \tag{6.4}$$

Step 3: solve the simultaneous equations. Equations (6.3) and (6.4) are easily solved to give

$$f(x) = -\frac{K}{2}, \quad g(x) = +\frac{K}{2} \quad \text{BUT ONLY FOR } x > 0 \text{ !}$$

from which we get

$$\begin{aligned} f(x-ct) &= -\frac{K}{2} && \text{for} \quad x - ct > 0 \\ g(x+ct) &= \frac{K}{2} && \text{for} \quad x + ct > 0 \quad (\text{ALWAYS TRUE}) \end{aligned}$$

So, for $x > ct$ we have

$$u(x,t) = f(x-ct) + g(x+ct) = 0$$

BUT we still have to solve for $f(x-ct)$ for $x < ct$:

Step 4: using the initial condition. We have $u(0,t) = \sin(\omega t)$. Hence

$$\begin{aligned} \sin(\omega t) &= u(0,t) = f(-ct) + g(ct) \\ &= f(-ct) + \frac{K}{2}, \end{aligned}$$

from which we get that $f(-ct) = \sin(\omega t)$, from which we note that $-ct < 0$. Hence, if we let $z = -ct$, we have

$$f(z) = \sin\left(\frac{-\omega}{c} z\right) - \frac{K}{2},$$

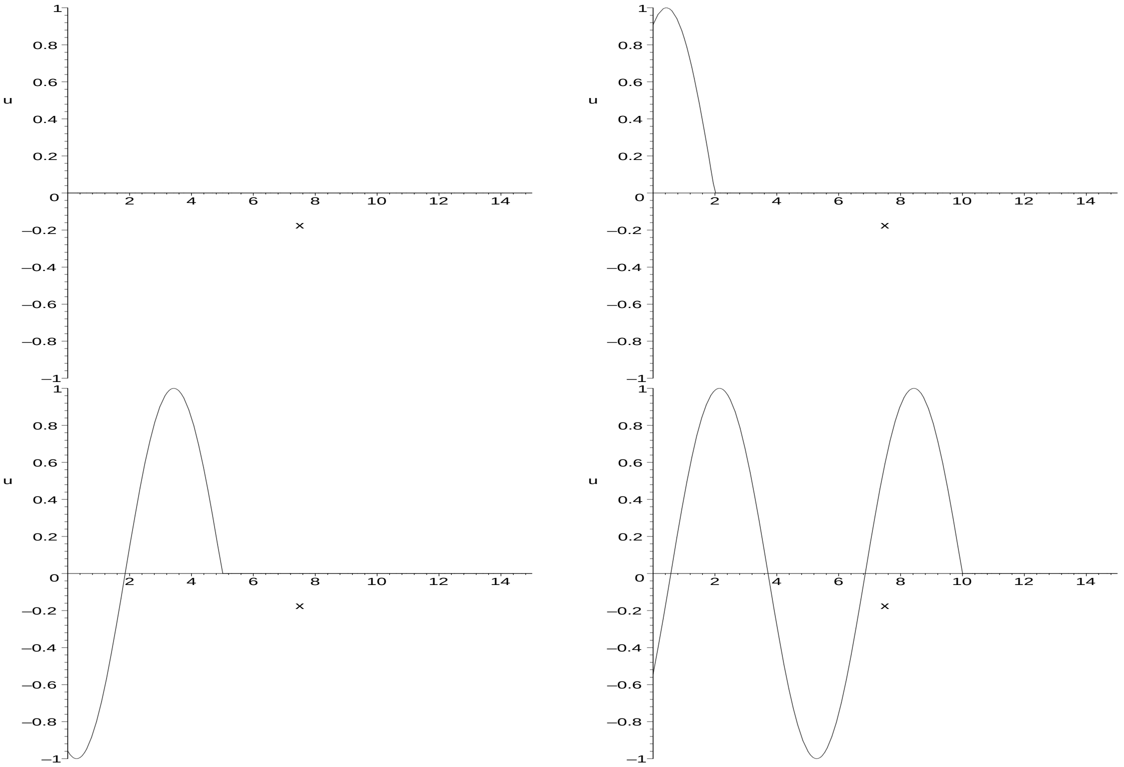
So, for $u < 0$ we have the solution

$$\begin{aligned} u(x,t) &= f(x-ct) + g(x+ct) \\ &= \sin\left(-\frac{\omega}{c}[x-ct]\right) - \frac{K}{2} + \frac{K}{2} \\ &= -\sin\left(\omega\frac{x-ct}{c}\right) \end{aligned}$$

Step 5: recombine to get general solution. So, the general solution is

$$u(x,t) = \begin{cases} -\sin\left(\omega\frac{x-ct}{c}\right) & x < ct \\ 0 & \text{otherwise} \end{cases}$$

Step 6: plot the solution profile. Plots for $ct = 0$, 2, 5 and 10



Note that this technique extends to equations with arbitrary initial conditions $u(x,0) = F(x)$ and $u_t(x,0) = G(x)$. Also we can have a boundary condition which, rather than specify a boundary value of $u(0,t) = A(t)$, like the previous example, specifies a condition on the normal derivative $u_x(0,t) = B(t)$

Worked example 6.3 *Use D'Alembert's method to find the general solution of the wave equation*

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le \infty, \quad t \ge 0,$$

subject to the initial conditions

$$u(x,0) = F(x), \qquad u_t(x,0) = G(x),$$

and boundary condition

$$u_x(x,0) = B(t)$$

Find the specific solution in the case that $F(x) = e^{-x}\sin x$*,* $G(x) = 0$ *and* $B(t) = \cos\omega t$*.*

6.4 Summary

- General solution to the wave equation is
$$u(x,t) = f(x-ct) + g(x+ct)$$
for arbitrary functions f and g.
- For wave equation on infinite domain $-\infty < x < \infty$, initial conditions $u(x,0) = F(x)$ and $u_t(x,t) = G(x)$ specify solution completely:
$$u(x,t) = \frac{1}{2}[F(x-ct) + G(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$
- For semi-infinite domain $0 \le x < \infty$, this solution works for $x > ct$. For $x < ct$, we have to use the boundary condition which is either of the form
$$u(0,t) = A(t), \qquad \text{or} \quad u_x(0,t) = B(t)$$

Note, maple commands to produce animated solutions:

```
[> restart: with(plots):
[> f:=y->piecewise(y<-1, 0, y<0,1+y,y<1, 1-y, y> 1, 0);
[> fb:=t->subs(c=1,(1/2)*f(x+c*t)+(1/2)*f(x-c*t));
[> animate(fb(t),x=-6..6,t=0..6,frames=200,numpoints=200);
[> g:=t->piecewise(x<c*t,-sin((x-c*t)/c),x>c*t,0): fc:=t->subs(c=1,g(t));
[> animate( fc(t),x=0..20,t=1..15,frames=200,numpoints=1000);
```