EMaII LSPDEs

Example Sheet 5: Inhomogeneous equations and d'Alembert

2009

1(a) A heated rod of length L is initially held at constant temperature T_0. At $t = 0$, the end $x = 0$ is cooled to T_1 and the end $x = L$ is heated to T_2. Show that the temperature in the rod $u(x,t)$ satisfies

$$u_t = \alpha^2 u_{xx} \qquad 0 \le x < L, \quad 0 \le t < \infty,$$

$u(0,t) = T_1$, $u(L,t) = T_2$, $u(x,0) = T_0$, where α^2 is a thermal conduction constant

(b) Use the principle of linear superposition to express the solution as a linear combination of $u = v + w$ where $v(x)$ solves the spatial boundary conditions.

(c) Hence show that $w(x,t)$ satisfies the same PDE but with boundary conditions $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = a + bx$, where $a = T_0 - T_1$, $b = (T_1 - T_2)/L$.

(d) Show that the Fourier sine series expansion of the function $f(x) = a + bx, \qquad 0 \le x < L$ has coefficients

$$b_n = \frac{2[(a+bL)(-1)^n - a]}{n\pi}.$$

(e) Hence solve the PDE for $u(x,t)$

2(a) A thin square conducting place of side length one unit is held at zero charge $u = 0$ on side $x = 0$ and $y = 0$. On the sides $x = 1$ and $y = 1$, the electric charge is maintained at Cy and Cx respectively, where C is a constant. The steady state charge distribution is therefore given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x < 1, \quad 0 \le y < 1$$

$u(0,y) = 0$, $u(1,y) = Cy$, $u(x,0) = 0$, $u(x,1) = Cx$. Use the principle of linear superposition to turn this into a sum of two problems, each with a single inhomogeneous boundary condition.

(b) Hence, use the separation of variables method to show the solution for the charge at position (x,y) in the plate is

$$u(x,y) = \frac{2C}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \sinh(n\pi)} [\sin(n\pi x)\sinh(n\pi y) + \sinh(n\pi x)\sin(n\pi y)].$$

3 Use d'Alembert's solution of the wave equation to solve the following problems on the infinite domain $-\infty < x < \infty$ In each case sketch the solution for $ct = 0$, $ct = 1$ and $ct = 10$.

(a)

$$u_{tt} = c^2 u_{xx}, \qquad \text{subject to} \quad u(x,0) = e^{-x^2}\sin(x), \quad u_t(x,0) = 0\,,$$

(b)

$$u_{tt} = c^2 u_{xx}, \qquad \text{subject to} \quad u(x,0) = \frac{1}{x^2} \quad u_t(x,0) = e^{-|t|}\cos t\,.$$

4 (not examinable) This example is aimed to show how the d'Alembert solution arises and how it is a specific example of the method of characteristics

(a) Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

Make the change of variables

$$v(x,t) = x - ct, \quad w(x,t) = x + ct\,.$$

Use the chain rule for partial differentiation to express u_{tt} and u_{xx} in terms of derivatives with respect to v and w.

(b) Hence show that the wave equation (1) when expressed in the new co-ordinates is

$$\frac{\partial^2 u}{\partial v \partial w} = 0\,. \tag{2}$$

(c) Consider the equation (2). Integrate once with respect to v. Then integrate once with respect to w. Hence show that the general solution to (2) can be expressed in the form

$$u(v,w) = f(v) + g(w)\,,$$

for arbitrary functions f and g. Hence derive the d'Alembert solution

$$u(x,t) = f(x - ct) + g(x + ct)\,.$$

The variables v and w are called the characteristic directions for the PDE, which are the directions in space and time in which information flows (forward travelling and backward travelling waves).