Q1, Sheet 3

$$\frac{dx}{dt} = \underbrace{\sqrt{\frac{xt}{x^2+t^2}}}_{f(x,t)}, \qquad x(0) = 1.$$

i.e. $x_0 = 1$, $t_0 = 0$

(a) $h = 0.1$. $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.3$

$$x_1 = x_0 + hf(x_0, t_0) = 1 + 0.1\sqrt{\frac{1\times 0}{1^2+0^2}}$$
$$= 1.$$

$$x_2 = x_1 + hf(x_1, t_1) = 1 + 0.1\sqrt{\frac{1\times 0.1}{1^2+0.1^2}}$$

$$= 1.031465839$$

$$x_3 = x_2 + hf(x_2, t_2) = 1.031465839 + 0.1\sqrt{\frac{1.0314\ldots\times 0.2}{1.0314\ldots^2+0.2^2}}$$

$$\underline{\underline{\hat{x}(0.3) = x_3 = 1.07694648}}$$

(b) $h = 0.05$, $t_0 = 0$, $t_1 = 0.05$, $t_2 = 0.1$, $t_3 = 0.15$, $t_4 = 0.2$, $t_5 = 0.25$, $t_6 = 0.3$

Starts to get a little tedious.

$$x_1 = x_0 + hf(x_0, t_0) = 1 + 0.05\sqrt{\frac{1\times 0}{1^2+0^2}} = 1.$$

$$x_2 = x_1 + hf(x_1, t_1) = 1 + 0.05\sqrt{\frac{1\times 0.05}{1^2+0.05^2}} = 1.011166391$$

$$x_3 = x_2 + hf(x_2, t_2) = 1 + 0.05\sqrt{\frac{1.0111\ldots\times 0.1}{1.0111\ldots^2+0.1^2}} = 1.026813901$$

$$x_4 = x_3 + hf(x_3, t_3) = \ldots\ldots = 1.045723597$$

$$x_5 = x_4 + hf(x_4, t_4) = \ldots\ldots = 1.067200687$$

$$\hat{x}(0.3) = x_6 = x_5 + hf(x_5, t_5) = \ldots\ldots = \underline{\underline{1.090762902}}$$

Latter answer should be more accurate: decreasing step size improves accuracy!

Q2, Sheet 3

$$\frac{dx}{dt} = \frac{x}{2\sqrt{t+x}}, \qquad x(0)=1.$$

Euler method, $h=0.1$. So $f(x,t) = \dfrac{x}{2\sqrt{t+x}}$

$t_0=0,\ t_1=0.1,\ t_2=0.2,\ t_3=0.3$

$x_0=1.$

$$x_1 = \hat{x}(t_1) = x_0 + hf(x_0,t_0)$$

$$= 1 + \frac{0.1\times 1}{2\sqrt{0+1}} = 1.05$$

$$x_2 = \hat{x}(t_2) = x_1 + hf(x_1,t_1)$$

$$= 1.05 + \frac{0.1\times 1.05}{2\sqrt{0.1+1.05}} = 1.098956502$$

$$x_3 = \hat{x}(t_3) = x_2 + hf(x_2,t_2) = 1.0989\ldots + \frac{0.1\times 1.098\ldots}{\sqrt{0.2+1.098\ldots}}$$

$$= \underline{\underline{1.195380070}}$$

Q3, Sheet 3

$\dot{x} = x, \quad x(0) = 1 \quad$ i.e. $f(x) = x$.

(i) $h = 0.1, \quad x_0 = 1, \quad t_0 = 0, t_1 = 0.1, t_2 = 0.2, t_3 = 0.3$

<u>explicit Euler</u>

$$x_{n+1} = x_n + h f(x_n) = (1+h)x_n.$$

$$x_1 = \hat{x}(0.1) = 1.1 \times x_0 = 1.1$$

$$x_2 = \hat{x}(0.2) = 1.1 \times x_1 = 1.21$$

$$x_3 = \hat{x}(0.3) = 1.1 \times x_2 = \underline{\underline{1.331}}$$

(ii) <u>Implicit Euler</u>.

$$x_{n+1} = x_n + h f(x_{n+1}), \qquad x_{n+1} = \frac{x_n}{1-h}$$

$$x_1 = \frac{x_0}{0.9} = \frac{1}{0.9} = 1.111\ldots$$

$$x_2 = \frac{x_1}{0.9} = 1.234567901\ldots$$

$$x_3 = \frac{x_2}{0.9} = \underline{\underline{1.371742112}}.$$

(iii) <u>Trapezoidal</u>

$$x_{n+1} = x_n + \frac{h}{2} f(x_n) + \frac{h}{2} f(x_{n+1})$$

So

$$x_{n+1} = \frac{\left(1 + \frac{h}{2}\right)}{\left(1 - \frac{h}{2}\right)} x_n = \left(\frac{1.05}{0.95}\right) x_n$$

$$x_1 = \frac{1.05}{0.95} = 1.105263158$$

$$x_2 = \left(\frac{1.05}{0.95}\right) x_1 = 1.221606648$$

$$x_3 = \left(\frac{1.05}{0.95}\right) x_2 = \underline{\underline{1.350196822}}$$

Exact answer $\quad \underline{\underline{e^{0.3} = 1.349858808}}$

Trapezoidal is much the best since its global accuracy is second order. The other methods are only first order accurate.

Q5, Sheet 3.

Each step of predictor-corrector for nonautonomous problem is:

$$\begin{cases} y_{n+1} = x_n + hf(x_n, t_n) \\ x_{n+1} = x_n + \frac{h}{2}f(x_n, t_n) + \frac{h}{2}f(y_{n+1}, t_{n+1}) \\ t_{n+1} = t_n + h. \end{cases}$$

$\frac{dx}{dt} = \sqrt{\{xt+t\}}$, i.e. $f(x,t) = \sqrt{\{xt+t\}} = \sqrt{\{t(1+x)\}}$

(a) $h=0.1$, $t_0=1$, $t_1=1.1$, $t_2=1.2$. $x_0=2$.

$$y_1 = x_0 + hf(x_0, t_0)$$
$$= 2 + 0.1\sqrt{\{2\times 1+1\}} = 2 + 0.1\sqrt{3} = 2.17320508$$

$$x_1 = x_0 + \frac{h}{2}f(x_0,t_0) + \frac{h}{2}f(y_1,t_1)$$
$$= 2 + 0.05\sqrt{3} + 0.05\sqrt{1.1(1+2.1732\ldots)} = 2.180017282.$$

Note correction is quite small

$$y_2 = x_1 + hf(x_1, t_1)$$
$$= 2.180017\ldots + 0.1\sqrt{1.1\,(1+2.1800\ldots)}$$
$$= 2.3670472\ldots$$

$$x_2 = x_1 + \frac{h}{2}f(x_1,t_1) + \frac{h}{2}f(y_2,t_2)$$
$$= 2.18\ldots + 0.05\sqrt{1.1(1+2.18\ldots)} + 0.05\sqrt{1.2(1+2.367\ldots)}$$
$$= \underline{\underline{2.374036677}} = X_1$$

Computation with other step sizes becomes very tedious!

Q5, Sheet 3 continued

(b) $h = 0.05$. 4 steps to get from $t_0 = 1$ to $t = 1.2$.

Answers only

$y_1 = 2.08660254\ldots$, $t_1 = 1.05$, $x_1 = 2.088307746$,

$y_2 = 2.178345559\ldots$, $t_2 = 1.1$, $x_2 = 2.180071841$,

$y_3 = 2.273587602\ldots$, $t_3 = 1.15$, $x_3 = 2.275336331$.

$y_4 = 2.372375457$, $t_4 = 1.2$, $\underline{\underline{x_4 = 2.374147858}} = X_2$

(c) $h = 0.025$, 8 steps required.

Now it gets realy tedious!

$y_1 = 2.043301270$, $t_1 = 1.025$, $x_1 = 2.043727857$

$y_2 = 2.087885395$, $t_2 = 1.05$, $x_2 = 2.088314540$

$y_3 = 2.1333\ldots$ $t_3 = 1.075$, $x_3 = 2.133765293$

etc. etc.

$y_8 = 2.373729520$, $t_8 = 1.2$, $\underline{\underline{x_8 = 2.374175806}} = X_3$

$$\left.\begin{aligned} |X_2 - X_1| &= 0.000111181 \\ |X_3 - X_2| &= 0.000027948 \end{aligned}\right\} \text{roughly a factor of 4 difference.}$$

Indeed, since predictor-corrector has $O(h^2)$ global error, we would expect error to go down by a factor of 4 if h is halved. (Compare with theory of extraction of rate of convergence from a numerically prescribed sequence.)

Sheet 3, Q6

$\dot{x} = 4x - x^3, \quad x(0)=1. \qquad$ i.e. $f(x) = 4x - x^3$

(i) $h=0.1, \quad t_1=0.1, t_2=0.2, t_3=0.3, t_4=0.4, t_5=0.5$

$x_1 = \hat{x}(0.1) = x_0 + hf(x_0) = 1 + 0.1(4-1) = 1.3$

$x_2 = \hat{x}(0.2) = x_1 + hf(x_1) = 1.3 + 0.1(4\times1.3 - 1.3^3) = 1.6003$

likewise, $x_3 = 1.830589557$

$x_4 = 1.949384179$

$x_5 = 1.988352627$

(ii) $h=0.2, \quad t_1=0.2, t_2=0.4, t_3=0.6, t_4=0.8, t_5=1.0$

$x_1 = \hat{x}(0.2) = x_0 + hf(x_0) = 1 + 0.2(4-1) = 1.6$

$x_2 = \hat{x}(0.4) = x_1 + hf(x_1) = 2.0608$

$x_3 = 1.959039081$

$x_4 = 2.022576940$

$x_5 = 1.985839873.$

(iii) $h=0.5$, omit working for brevity.

$x_1 = 2.5, \quad x_2 = -0.3125, \quad x_3 = -0.9222412110,$

$x_4 = -2.374527254, \quad x_5 = -0.429338652$

(iv) $h=1.0. \quad x_1 = 4, \; x_2 = -44.0, \; x_3 = 84964.0, \; x_4 = -6.133.. \times 10^{14},$

$x_5 = 2.307... \times 10^{44}$

<u>Comments</u> : This ODE can be solved explicitly in fact.

Exact soln is $\quad x(t) = \dfrac{2e^{4t}}{\sqrt{3+e^{8t}}} \quad \to 1$ as $t \to \infty$

(i) has correct qualitative behaviour.

(ii) is roughly ok, but convergence to limit is oscillatory & shouldn't be

(iii) Completely wrong.

(iv) Wrong and blowing up to ∞.

NUMERICAL INSTABILITY.

Sheet 3, Q7

$\frac{dx}{dt} = x^2$, $\quad x(0) = 1$.

$\int \frac{1}{x^2}\,dx = \int dt$, $\quad -\frac{1}{x} = t + c$, $\quad c = -1$ fits initial data.

$\therefore$ Explicit exact soln is $\quad \underline{\underline{x = \frac{1}{1-t}}} \quad \to +\infty$ as $t \to 1-$.

Can the Euler method mimic this behaviour?

<u>No</u>, not with a fixed time step.

With a fixed time step, we will reach $t = 1$ with a finite number of steps, and x value will be finite.

Can only have blow up in numerical scheme as # steps $\to$ infinity.

Sheet 3, Q8.

$\frac{dx}{dt} = \lambda x$, $\lambda < 0$. Explicit exact soln $x = ce^{\lambda t} \to 0$ as $t \to \infty$.

(a) explicit Euler method is not A_0 stable.

$$x_{n+1} = x_n + hf(x_n) = (1+\lambda h)x_n.$$

Solution of difference equation is $x_n = c(1+\lambda h)^n x_0$.

For stability, require $-1 < 1+\lambda h < +1$.

Since $\lambda < 0$, $h > 0$, require $-1 < 1+\lambda h$.

i.e. $(-\lambda)h < 2$, i.e. $\underline{\underline{h < \frac{2}{|\lambda|}}}$ is max. time step.

(b) Midpoint method is off syllabus. In fact, it is unconditionally stable.

(c) Trapezoidal method is A_0 stable, i.e. stable whatever the time step.

(d) Second order predictor-corrector.

$$y_{n+1} = x_n + hf(x_n) = (1+\lambda h)x_n.$$

$$x_{n+1} = x_n + \frac{h}{2}f(x_n) + \frac{h}{2}f(y_{n+1}) = x_n + \frac{\lambda h}{2}x_n + \frac{\lambda h}{2}(1+\lambda h)x_n$$
$$= \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right)x_n.$$

For stability, we require

$$-1 < 1 + \lambda h + \frac{\lambda^2 h^2}{2} < +1.$$

i.e. $\frac{\lambda^2 h^2}{2} + \lambda h + 2 > 0$, OR $\lambda h + \frac{\lambda^2 h^2}{2} < 0$

NB "$b^2 - 4ac$" $= 1 - 4.2.\frac{1}{2}$
$= -3 < 0$.

so this is always true

$\Rightarrow \underbrace{\lambda h}_{<0}\left(1 + \frac{\lambda h}{2}\right) < 0$

i.e. $1 + \frac{\lambda h}{2} > 0$ required

i.e. $h < \frac{2}{|\lambda|}$

Sheet 3, Q8 continued.

(e)
$$y_1 = x_n$$
$$y_2 = x_n + \frac{h}{2} f(y_1) = x_n + \frac{\lambda h}{2} x_n = \left(1 + \frac{\lambda h}{2}\right) x_n$$
$$y_3 = x_n + \frac{h}{2} f(y_2) = x_n + \frac{\lambda h}{2}\left(1 + \frac{\lambda h}{2}\right) x_n$$
$$= \left(1 + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{4}\right) x_n.$$
$$y_4 = x_n + h f(y_3) = x_n + \lambda h \left(1 + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{4}\right) x_n.$$
$$y_4 = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right) x_n$$
$$x_{n+1} = x_n + \frac{h}{6}\left[f(y_1) + 2f(y_2) + 2f(y_3) + f(y_4)\right]$$
$$x_{n+1} = \left[1 + \frac{\lambda h}{6} + \frac{\lambda h}{3}\left(1 + \frac{\lambda h}{2}\right) + \frac{\lambda h}{3}\left(1 + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{4}\right) + \frac{\lambda h}{6}\left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{4}\right)\right] x_n$$
$$= \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}\right) x_n.$$

So require $$-1 < 1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24} < +1$$

$$\underbrace{2 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24}}_{} > 0$$

it may be shown that this is true for all (λh).

$$\lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6} + \frac{\lambda^4 h^4}{24} < 0$$

$$\underbrace{\lambda h}_{<0}\underbrace{\left(1 + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{6} + \frac{\lambda^3 h^3}{24}\right)}_{\text{so need this } >0.} < 0$$

$h \cancel{<} \frac{2}{|\lambda|}$ is certainly ok

$h > \frac{3}{|\lambda|}$ is certainly not ok.

$$g(u) = 1 + \frac{u}{2} + \frac{u^2}{6} + \frac{u^3}{24}$$

$$g(-2) = 1 - 1 + \frac{4}{6} - \frac{8}{24} > 0$$

$$g(-3) = 1 - \frac{3}{2} + \frac{9}{6} - \frac{27}{24} < 0$$