Engineering Mathematics 2: Numerical Analysis
Exercise Sheet 3

C. Colijn

For real world applications, one would not dream of solving numerical ODE problems by hand / with a calculator. Unfortunately, in the exam, this is exactly what you might have to do, although you would never be asked to apply more than three or four steps of a method.

You might find some of the numerical examples below tedious as they require e.g. 5 steps or more. Why not write a short piece of computer code to attack these examples?

Euler Method

1. Use the Euler method with $h = 0.1$ to construct an approximate solution of the differential equation problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{xt}{x^2+t^2}}\,, \qquad x(0) = 1$$

so as to find an approximation to $x(0.3)$. Repeat the calculation with $h = 0.05$. Which answer would you expect to be more accurate?

2. Find an approximation to $x(0.3)$ where $x(t)$ is the solution of the differential equation problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x}{2\sqrt{t+x}}\,, \qquad x(0) = 1,$$

using the Euler method with step size $h = 0.1$.

Trapezoidal method and implicit Euler

3. Find an approximation to $x(0.3)$, where $x(t)$ solves the differential equation problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x, \qquad x(0) = 1,$$

using (i) the Euler method, and (ii) the Trapezoidal method, and (iii) the implicit (i.e., backward) Euler method, in each case using step size $h = 0.1$. Compare with the exact answer $\mathrm{e}^{0.3}$. Which method gives the most accurate answer and why?

4. * The Trapezoidal method with step size h is applied to the solution of the *nonlinear* differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2.$$

   (a) Derive an algebraic equation in the form

$$F(x_{n+1}, h) = G(x_n, h),$$

   relating the approximation x_{n+1} at step $(n+1)$ to that at step n.

(b) Use the Trapezoidal method with $h = 0.2$ to find an approximation to $x(1)$ when initial data $x(0) = 1$ is provided. (*Hint*: solve the algebraic equation of part (a) exactly, using the well-known formula. At each step choose the most natural root.)

(c) Perform one step of the Trapezoidal method with $h = 0.2$, to approximate $x(0.2)$, where $x(t)$ solves the differential equation problem posed in parts (a) and (b). In this case, use 3 Newton-Raphson iterations to solve the algebraic equation derived in part (a). (*Hint*: you may use x_0 as your initial guess of the value of x_1.)

(d) Repeat the whole question for the implicit (i.e., backward) Euler method rather than the Trapezoidal method.

Predictor-Corrector method

5. Approximate $x(1.2)$ using the second order predictor-corrector method (Euler + Trapezoidal) for the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{xt + t}, \qquad x(1) = 2,$$

using (a) $h = 0.1$, (b) $h = 0.05$, (c) $h = 0.025$. Let X_1, X_2 and X_3 be the approximations to the $x(1.2)$ using $h = 0.1$, 0.05 and 0.025 respectively. Compute the ratio of $|X_2 - X_1|$ to $|X_3 - X_2|$. Show theoretically that this ratio should tend to 4:1 as h tends to zero.

Stability

6. Apply five steps of the Euler method to the approximate solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - x^3, \qquad x(0) = 1,$$

with step size (i) $h = 0.1$, (ii) $h = 0.2$, (iii) $h = 0.5$, and (iv) $h = 1.0$. Comment on your results.

7. Show by explicit solution that the differential equation problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2, \qquad x(0) = 1,$$

has a solution which "blows up" to infinity at $t = 1$. Can the Euler method approximation mimic this behaviour?

8. A numerical ODE solver is called A_0 stable if when applied to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda x, \qquad \text{where } \lambda < 0,$$

it gives a numerical solution decaying to zero, for any step size h. Identify which of the following methods are A_0 stable, and for those which are not, derive conditions on h and λ for the numerical solution to decay: (a) Euler method, (b) Midpoint method, (c) Trapezoidal method, (d) second order Predictor-Corrector method, (e) fourth order Runge Kutta method.

9. The differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -12x + \frac{3}{2}t^2 - 12t + 6t^3 - 1, \qquad x(0) = 0,$$

is an example of a so-called stiff equation. Estimate $x(1)$ using a step size $h = 0.1$ with (a) the midpoint method, (b) the second order predictor-corrector method. Compare each solution with the analytical solution $y = (t^3/2) - t$. Despite both being second-order methods, which gives the best results? Why?

Fourth order Runge-Kutta method

10. Estimate $x(t)$ for $t = 0.2$, 0.4 and 0.6, where x satisfies the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{x+t}, \qquad x(0) = 2,$$

by using the fourth-order Runge-Kutta method with step $h = 0.2$.

Systems of equations *

11. Use the Euler method with step size $h = 0.1$ to estimate $x(0.5)$ and $y(0.5)$, where $x(t)$, $y(t)$ solve the pair of simultaneous first-order equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy + t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - t,$$

with initial data $x(0) = 0$ and $y(0) = 1$.

12. Consider the approximate numerical solution of the differential equation system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = +x - 100y.$$

(Note that this system may be re-written as a scalar second order equation for either x or y alone.) Write down difference equations corresponding to the discretization of this linear system when

(a) the Euler method,

(b) the Trapezoidal method

is applied. By analysing the eigenvalues of the original system, and solving each set of difference equations explicitly, derive stability criteria for each method. Which is superior?