EMaII PDEs

Example Sheet 5: Inhomogeneous equations and d'Alembert

2007

1(a) Let $u = v(x, y) + w(x, y)$, where w solves the homogeneous problem. In this case we shall

$$\text{try} \quad v(x, y) = e^x + e^y \ .$$

Note that $v_{xx} = e^x$ and $v_{yy} = e^y$. Hence

$$v_{xx} + v_{yy} = e^x + e^y,$$

and so v is a particular integral of the inhomogeneous problem.

Now, consider the boundary conditions.using $u(x, y) = v(x, y) + w(x, y)$

$$\begin{aligned}
0 = u(0, y) = 1 + e^y + w(0, y) \quad &\Rightarrow \quad w(0, y) = -1 - e^y \\
0 = u(a, y) = e^a + e^y + w(a, y) \quad &\Rightarrow \quad w(a, y) = -e^a - e^y \\
0 = u(x, 0) = e^x + 1 + w(x, 0) \quad &\Rightarrow \quad w(x, 0) = -1 - e^x \\
0 = u(x, b) = e^x + e^b + w(x, b) \quad &\Rightarrow \quad w(x, b) = -e^b - e^x
\end{aligned}$$

1(b) We set $u(x, t) = v(x) + w(x, t)$, where we

$$\text{try} \quad v(x) = A \sin(x).$$

Substituting this form into the PDE $v_{tt} - c^2 v_{xx} = \sin(x)$ gives

$$-c^2 A \sin(x) = \sin(x),$$

hence $A = -1/c^2$. So we have

$$v(x) = -(1/c^2)\sin(x), \quad w_{tt} - c^2 w_{xx} = 0.$$

Now, let us consider the boundary conditions. We get

$$\begin{aligned}
0 = u(0, t) = w(0, t) \quad &\Rightarrow \quad w(0, t) = 0 \\
0 = u(\pi, t) = 0 + w(a, y) \quad &\Rightarrow \quad w(a, y) = 0 \\
0 = u(x, 0) = -(1/c^2)\sin(x) + w(x, 0) \quad &\Rightarrow \quad w(x, 0) = \sin(x)/c^2 \\
0 = u_t(x, b) = -(1/c^2)\cos(x) + w_t(x, b) \quad &\Rightarrow \quad w_t(x, b) = \cos(x)/c^2
\end{aligned}$$

1(c) Let $u(x, t) = v(x, t) + w(x, t)$ and

$$\text{try} \quad v(x, t) = Ax^2 t + Ct^2.$$

Substituting this form into the PDE we get

$$Ax^2 + 2Ct - 2A\alpha^2 t = x^2.$$

The coefficient of x^2 gives $A = 1$, and the coefficient of t gives $2C = 2A\alpha^2$, so $C = \alpha^2$. Hence, we get

$$v(x,t) = x^2 t + \alpha^2 t^2, \qquad \text{and} \quad w_t - \alpha^2 w_{xx} = 0.$$

Now, from the boundary conditions we obtain

$$\begin{aligned} 0 = u(x,0) = 0 + w(x,0) &\Rightarrow w(x,0) = 0 \\ 0 = u(0,t) = \alpha^2 t^2 &\Rightarrow w(0,t) = -\alpha^2 t^2 \\ 0 = u(L,t) = L^t + \alpha^2 t^2 + w(L,t) &\Rightarrow w(L,t) = -L^2 t - \alpha^2 t^2 \end{aligned}$$

2(a) Let $u(x,t)$ represent the temperature field at position x and time t within the bar. The fact that ends $x = 0$ and $x = L$ are held at constant temperatures T_1 and T_2 respectively implies

$$u(0,t) = T_1, \qquad u(L,t) = T_2.$$

The initial temperature is held constant at T_0, so

$$u(x,0) = T_0.$$

Finally, heat conduction within the bar, given a thermal condition constant α^2, satisfies the heat equation

$$u_t = \alpha^2 u_{xx} \qquad 0 \leq x < L, \quad 0 \leq t < \infty.$$

2(b) To solve this equation we start by letting $u = v(x) + w(x,t)$ where v is chosen to solve the spatial boundary values

$$v(0) = T_1, \quad v(L) = T_2.$$

Hence, we choose

$$v = T_1(1 - x/L) + T_2 x/L\,.$$

Note that such a function satisfies the PDE, because $v_t = 0$ by definition and $v_{xx} = 0$ since v is only linear in x.

2(c) Using the principle of linear superposition $w(x,t)$ satisfies the same PDE, but with different boundary conditions. To work out the new boundary conditions we set $u(x,t) = v(x) + w(x,t)$ and apply to the original boundary conditions. Thus:

$$\begin{aligned} T_1 = u(0,t) = T_1 + w(0,t) &\Rightarrow w(0,t) = 0 \\ T_2 = u(L,t) = T_2 + w(L,t) &\Rightarrow w(L,t) = 0 \\ T_0 = u(x,0) = T_1(1 - x/L) + T_2 x/L &\Rightarrow w(x,0) = (T_0 - T_1) + (x/L)(T_1 - T_2)\,. \end{aligned}$$

2(d) Now, if we let $T_0 - T_1 := a$ and $\frac{T_1 - T_2}{L} := b$ then, by definition we have

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L (a + bx)\sin(n\pi x L)\,dx \\ &= \frac{2}{L}\left(\left[-\frac{L(ax+b}{n\pi}\cos(n\pi x L)\right]_0^L + \int_0^L \frac{Lb}{n\pi}\cos(n\pi x L)\,dx\right) \\ &= \left[-\frac{2(ax+b}{n\pi}\cos(n\pi x L) + \frac{2Lb}{n^2\pi^2}\sin(n\pi x L)\right]_0^L \\ &= \left[\frac{(a+bL)}{2n\pi}\cos(n\pi)\right] - \left[\frac{2b}{n\pi}\cos(n\pi)\right] \\ &= \left[\frac{2(a+bL)}{n\pi}(-1)^n\right] - \left[\frac{2b}{n\pi}\right] \end{aligned}$$

Hence we have

$$u(x,t) = a + bx + \sum_{n=1}^{\infty} 2\frac{(a+bL)\pi^2(-1)^n - b}{n\pi} \sin(n\pi x L),$$

where $a = T_0 - T_1$ and $b = \frac{T_1 - T_2}{L}$.

3(a) We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x < 1, \quad 0 \le y < 1$$

$u(0,y) = 0$, $u(1,y) = Cy$, $u(x,0) = 0$, $u(x,1) = Cx$. Now, if we let $u(x,y) = v(x,y) + w(x,y)$, we can stipulate that both v and w should satisfy Laplace's equation, but that each one takes a different inhomogeneous boundary condition. So we can choose

$$v(0,y) = 0, \quad v(1,y) = Cy, \quad v(x,0) = 0, \quad v(x,1) = 0,$$

and

$$w(0,y) = 0, \quad w(1,y) = 0, \quad w(x,0) = 0, \quad w(x,1) = Cx.$$

3(b) Now we can use the separation of variables method for v and w separately. Starting with $v(x,y)$, we set

$$v(x,y) = X_1(x)Y_1(y).$$

Substituting this function into Laplace's equation and dividing by X_1Y_1 we obtain

$$\frac{X_1''}{X_1}(x) = -\frac{Y_1''}{Y_1}(y) = k_1^2.$$

Notice that we choose the constant k_1^2 to be positive because we want sinusoidal solutions in y in order to satisfy the inhomogeneous boundary condition $= Cy$. Thus we get the two separate ODEs

$$X_1'' = k_1 X_1, \qquad Y_1'' = -k_1 Y_1,$$

which have solution

$$X_1(x) = A\cosh(k_1 x) + B\sinh(k_1 x), \quad Y(y) = C\cos(k_1 y) + D\sin(k_1 y)$$

Applying the homogeneous boundary conditions for v we have

$$\begin{aligned}
0 = v(0,y) &\Rightarrow X_1(0)Y_1(y) = A\cosh(0)Y_1(y) &&\Rightarrow A = 0,\\
0 = v(x,0) &\Rightarrow X_1(x)Y_1(0) = X_1(x)C\cos(0) &&\Rightarrow C = 0\\
0 = v(x,1) &\Rightarrow X_1(x)Y_1(1) = X_1(x)D\sin(k_1) &&\Rightarrow k_1 = n\pi,
\end{aligned}$$

for some integer n. Hence we have

$$v(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y)\sinh(n\pi x),$$

where $CD = b_n$. Now, solving the inhomogeneous boundary condition, we have

$$Cy = v(1,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi y)\sinh(n\pi),$$

Hence, from the Fourier half-range sine series we have

$$b_n \sinh(n\pi) = 2\int_0^1 Cy\sin(n\pi y)dy$$

Solving this integral, we get

$$\begin{aligned}
b_n \sinh(n\pi) &= 2\int_0^1 Cy\sin(n\pi y)dy \\
&= 2\left(\left[-\frac{Cy}{n\pi}\cos(n\pi y)\right]_0^1 + \int_0^1 \frac{C}{n^2\pi^2}\cos(n\pi y)dy\right) \\
&= \left[-\frac{2Cy}{n\pi}\cos(n\pi y) + \frac{2C}{n^2\pi^2}\sin(n\pi y)\right]_0^1 \\
&= \left[-\frac{2C}{n\pi}\cos(n\pi)\right] - [0] \\
&= -\frac{2C(-1)^{n+1}}{n\pi}
\end{aligned}$$

Similarly, we can solve for $w(x,y)$. Note that w satisfies exactly the same PDE as v with the same boundary conditions BUT with x replaced by y and vice versa. Hence we have

$$w(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh(n\pi y),$$

where the b_n are exactly as above. Hence we get the general solution $u = v + w$:b

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2C(-1)^{n+1}}{n\pi\sinh(n\pi)}\left(\sin(n\pi x)\sinh(n\pi y) + \sinh(n\pi x)\sin(n\pi y)\right)$$

4(a) We start by stating the d'Alembert solution

$$u(x,t) = f(x-ct) + g(x+ct).$$

Posing the initial condition $u_t(x,0) = 0$, we find

$$0 = -cf'(x) + cg'(x), \quad \Rightarrow g(x) - f(x) = K.$$

for some constant K. The other initial condition, $u(x,0) = e^{-x^2}\sin(x)$, gives

$$f(x) + g(x) = e^{-x^2}\sin(x)$$

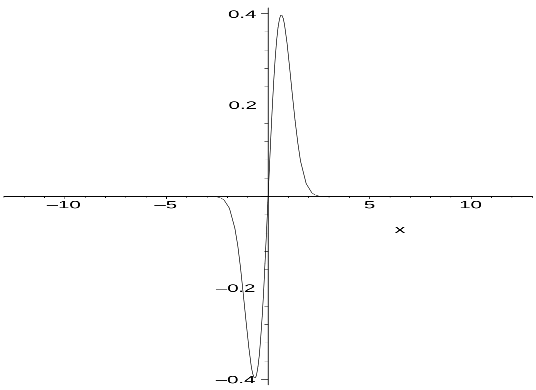
Hence we have

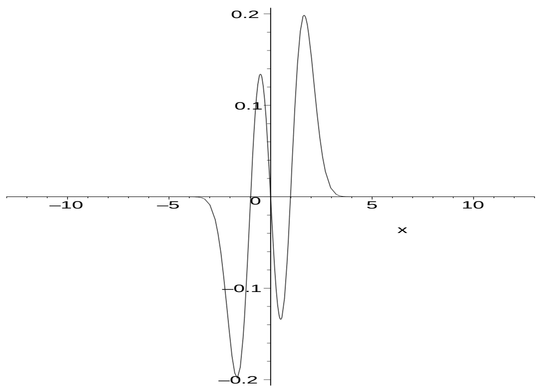
$$\begin{aligned}
g(x) &= \frac{K}{2} + \frac{1}{2}e^{-x^2}\sin(x) \\
f(x) &= -\frac{K}{2} + \frac{1}{2}e^{-x^2}\sin(x).
\end{aligned}$$

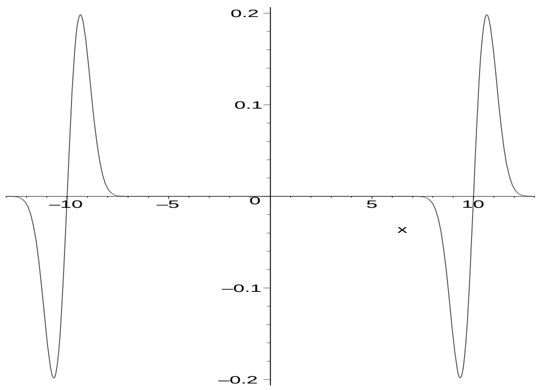
Which gives

$$u(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\left(e^{-(x+ct)^2}\sin(x+ct) - e^{-(x-ct)^2}\sin(x-ct)\right).$$

The solutions for $t = 0$, $t = 1$ and $t = 10$ with $c = 1$ are plotted below







4(b) Again we write down d'Alembert's solution

$$u(x,t) = f(x-ct) + g(x+ct)$$

and pose the boundary conditions. The initial velocity condition gives

$$e^{-|x|}\cos(x) = -cf'(x) + cg'(x).$$

Which implies that

$$g(x) - f(x) = \frac{1}{c}\int e^{-|x|}\cos(x)dx$$

Let $I = \int e^{-|x|}\cos(x)$. Then, integration by parts twice yields

$$\begin{aligned} I &= [-e^{-|x|}\sin(x)] + \int \text{sign}(x)e^{-|x|}\sin(x)dx \\ &= [-e^{-|x|}\sin(x)] + [\text{sign}(x)e^{-|x|}\cos(x)] - \int (\text{sign}(x))^2 e^{-|x|}\cos(x)dx \\ &= [-e^{-|x|}\sin(x) + \text{sign}(x)e^{-|x|}\cos(x)] - I\,. \end{aligned}$$

Hence,

$$I = \frac{1}{2}(\text{sign}(x)e^{-|x|}\cos(x) - e^{-|x|}\sin(x)) + K$$

and so

$$g(x) - f(x) = \frac{1}{2c}(\text{sign}(x)e^{-|x|}\cos(x) - e^{-|x|}\sin(x)) + K \tag{1}$$

The initial condition on position gives

$$u(x,0) = f(x) + g(x) = \frac{1}{1+x^2} \tag{2}$$

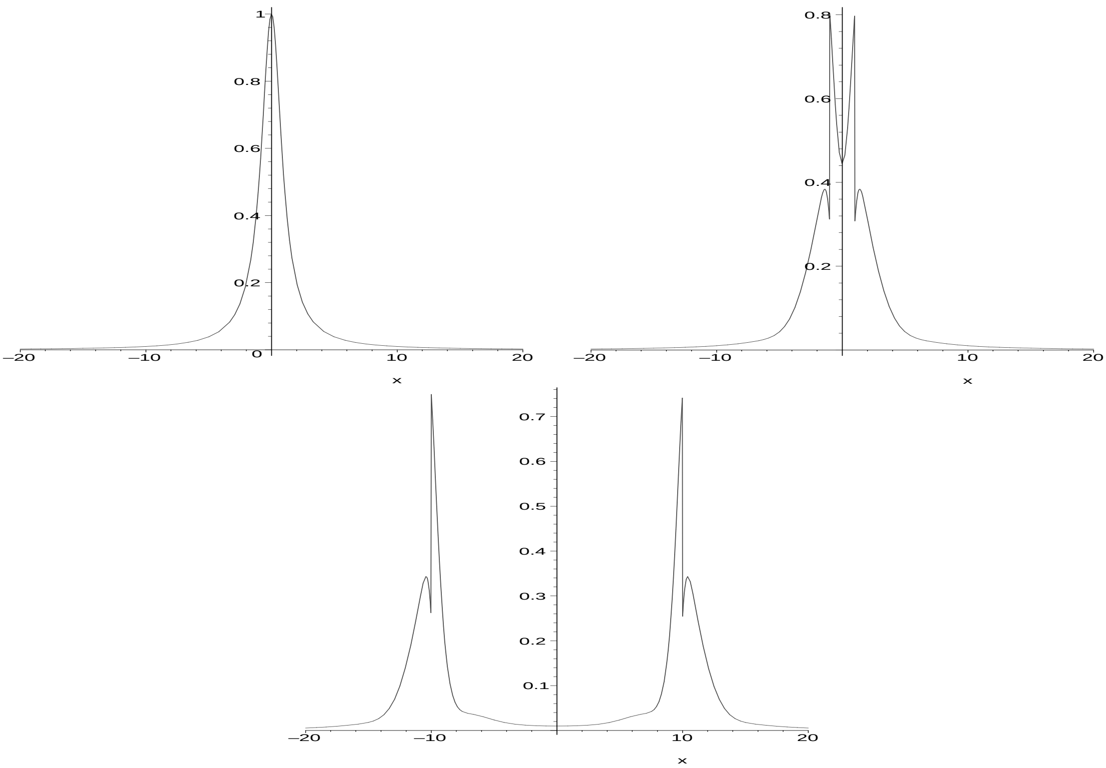
Solving the simultaneous equations (1) and (2) gives

$$\begin{aligned} g(x) &= \frac{1}{2(1+x^2)} + \frac{1}{4c}(\text{sign}(x)e^{-|x|}\cos(x) - e^{-|x|}\sin(x)) + K/2 \\ f(x) &= \frac{1}{2(1+x^2)} - \frac{1}{4c}(\text{sign}(x)e^{-|x|}\cos(x) - e^{-|x|}\sin(x)) - K/2. \end{aligned}$$

Hence $u(x,t) = f(x-ct) + g(x+ct)$ gives

$$\begin{aligned} u(x,t) &= \frac{1}{2[1+(x-ct)^2]} + \frac{1}{2(x+ct)^2} + \frac{1}{4c}(\text{sign}(x+ct)e^{-|x+ct|}\cos(x+ct) - e^{-|x+ct|}\sin(x+ct)) \\ &\quad -\frac{1}{4c}(\text{sign}(x-ct)e^{-|x-ct|}\cos(x-ct) - e^{-|x-ct|}\sin(x-ct)) \end{aligned}$$

The solutions for $t = 0$, $t = 1$ and $t = 10$ with $c = 1$ are plotted below



5(a) We start by writing down the d'Alembert solution

$$u(x,t) = f(x-ct) + g(x+ct).$$

Posing the initial velocity condition we get

$$e^{-x} = u_t(x,0) = -cf'(x) + cg'(x).$$

Hence,

$$g(x) - f(x) = -\frac{1}{c}e^{-x} + K, \qquad (3)$$

for some arbitrary constant K. Posing the initial condition on position we get

$$\frac{1}{x^2} = u(x,0) = f(x) + g(x).$$

Solving the simultaneous equations (3) and () we find

$$\begin{aligned} f(x) &= \frac{1}{2(1+x)^2} + \frac{1}{2c}e^{-x} - \frac{K}{2} \\ g(x) &= \frac{1}{2(1+x)^2} - \frac{1}{2c}e^{-x} + \frac{K}{2} \end{aligned}$$

both of which are only valid for $x > 0$. Hence we find

$$g(x+ct) = \frac{1}{2(1+x+ct)^2} - \frac{1}{2c}e^{-x-ct} + \frac{K}{2}$$

for all $x + ct > 0$ (which is true for all $t > 0$), and

$$f(x-ct) = \frac{1}{2(1+x-ct)^2} + \frac{1}{2c}e^{-x+ct} - \frac{K}{2}$$

for $x - ct > 0$.

Hence, it remains to solve for $f(x-ct)$ for $x < ct0$. Do do this we use the boundary condition at $t = 0$. Hence we obtain

$$\begin{aligned}\sin(\pi t) &= u(0,t) = f(-ct) + g(ct) \\ &= f(-ct) + \frac{1}{2(1+ct)^2} - \frac{1}{2c}e^{-ct} + \frac{K}{2},\end{aligned}$$

from which we note

$$f(-ct) = \cos(\pi t) - \frac{1}{2(1+ct)^2} + \frac{1}{2c}e^{-ct} - \frac{K}{2},$$

for $t > 0$. Letting $z = -ct$, we find

$$f(z) = \cos\left(\frac{\pi z}{c}\right) - \frac{1}{2(1-z)^2} + \frac{1}{2c}e^{z} - \frac{K}{2},$$

for an aritrary argument $z < 0$. Hence we obtain

$$f(x-ct) = \cos\left(\frac{\pi x - ct}{c}\right) - \frac{1}{2(1-x+ct)^2} + \frac{1}{2c}e^{x-ct} - \frac{K}{2},$$

for $x - ct < 0$.

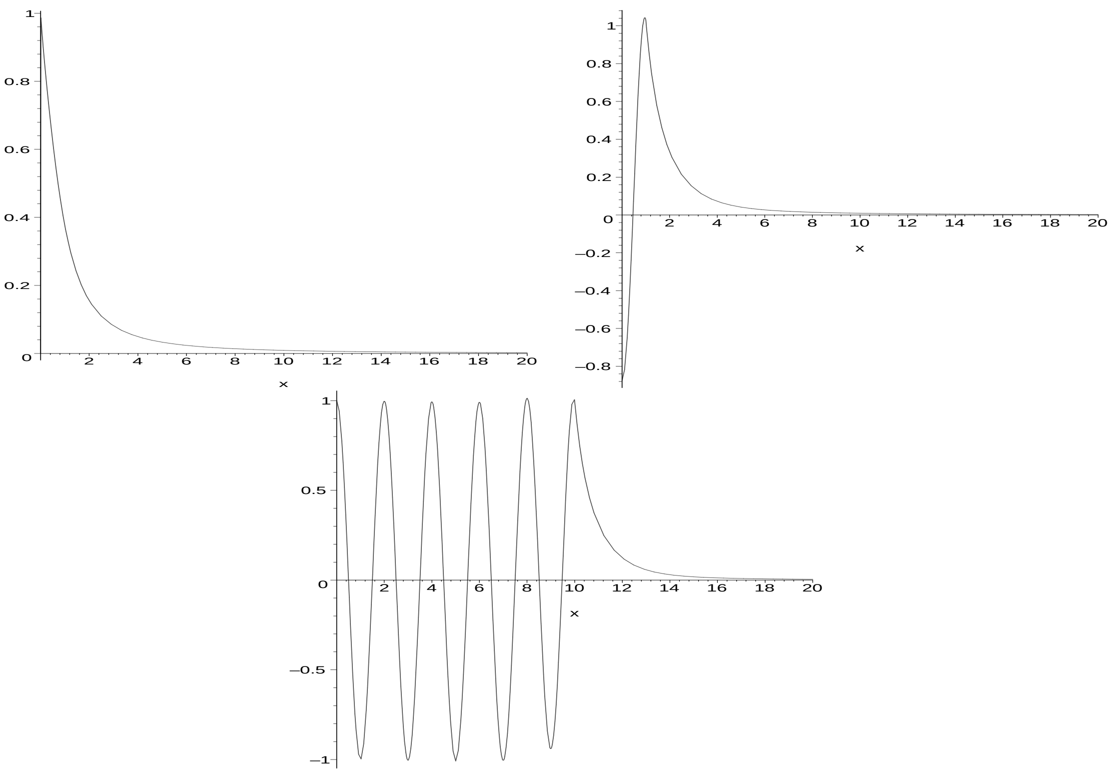
Adding f and g, we obtain

$$u(x,t) = \frac{1}{2(1+x+ct)^2} - \frac{1}{2c}e^{-x-ct} + \frac{1}{2(1+x-ct)^2} + \frac{1}{2c}e^{-x+ct}$$

for $x > ct$ and

$$u(x,t) = \frac{1}{2(1+x+ct)^2} - \frac{1}{2c}e^{-x-ct}\cos\left(\frac{\pi x - ct}{c}\right) - \frac{1}{2(1-x+ct)^2} + \frac{1}{2c}e^{x-ct}$$

for $x < ct$. The solution for $ct = 0$, $ct = 1$ and $ct = 10$ is plotted below



5(b) We start by writing down the d'Alembert solution

$$u(x,t) = f(x-ct) + g(x+ct).$$

Posing the initial velocity condition we get

$$e^{-x} = u_t(x,0) = -cf'(x) + cg'(x).$$

Hence,

$$g(x) - f(x) = -\frac{1}{c}e^{-x} + K, \tag{4}$$

for some arbitrary constant K. Posing the initial condition on position we get

$$0 = u(x,0) = f(x) + g(x).$$

Solving the simultaneous equations (4) and () we find

$$\begin{aligned} f(x) &= \frac{1}{2c}e^{-x} - \frac{K}{2}, \\ g(x) &= -\frac{1}{2c}e^{-x} + \frac{K}{2} \end{aligned}$$

both of which are only valid for $x > 0$. Hence we find

$$g(x+ct) = -\frac{1}{2c}e^{-x-ct} + \frac{K}{2},$$

for all $x + ct > 0$ (which is true for all $t > 0$), and

$$f(x-ct) = \frac{1}{2c}e^{-x+ct} - \frac{K}{2},$$

for $x - ct > 0$.

It remains to solve for $f(x-ct)$ for $x < ct0$. Do do this we use the boundary condition at $t = 0$. Thus, we obtain

$$\begin{aligned} \frac{1}{2} - \cos^2(2\pi t) &= u_x(0,t) = f'(-ct) + g'(ct) \\ &= f'(-ct) - \frac{1}{2c}e^{-ct}, \end{aligned}$$

from which we get

$$f'(-ct) = \frac{1}{2} - \cos^2(2\pi t) + \frac{1}{2c}e^{-ct}$$

Letting $z = -ct$ we find

$$f'(z) = \frac{1}{2} - \cos^2\left(\frac{2\pi z}{c}\right) + \frac{1}{2c}e^{z},$$

for any argument $z < 0$. Upon integration we find

$$f(z) = -\frac{c}{8\pi}\sin\left(\frac{4\pi z}{c}\right) + \frac{1}{2c}e^{z} + \text{const.}$$

Hence

$$f(x-ct) = -\frac{c}{8\pi}\sin\left(\frac{4\pi(x-ct)}{c}\right) + \frac{1}{2c}e^{x-ct} + \text{const.}.$$

We choose the constant equal to $-K/2$ in order to cancel the $+K/2$ in $g(x+ct)$.

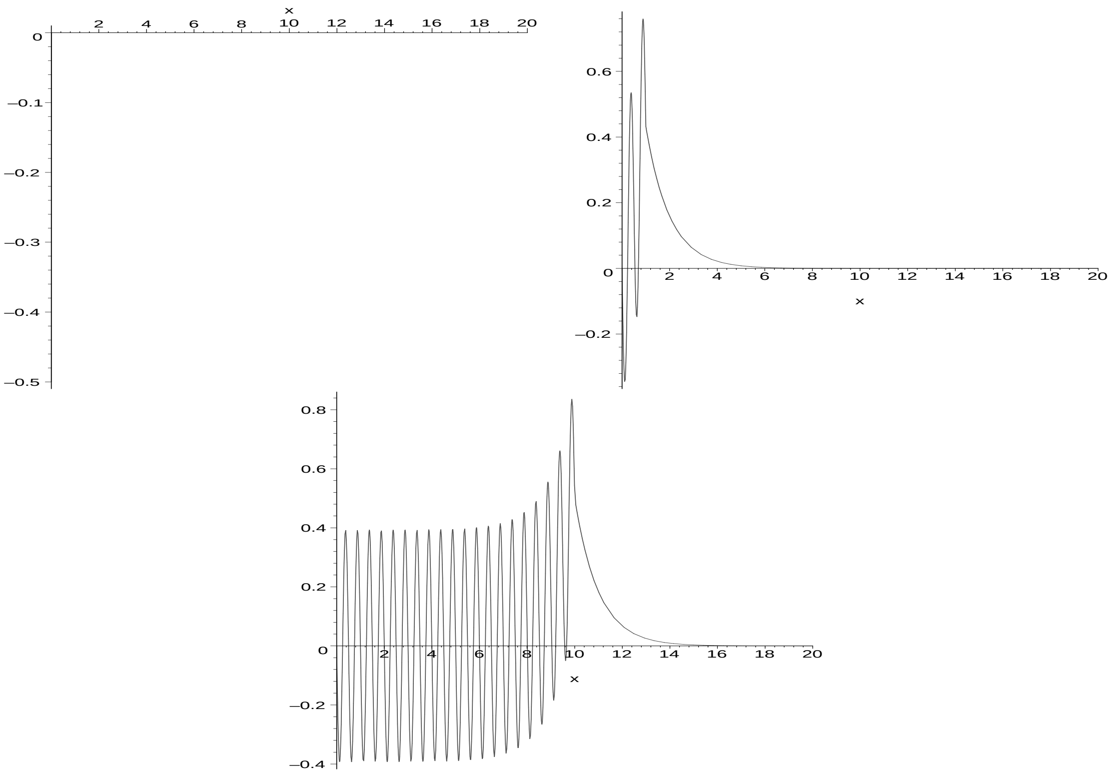
We get the general solution upon adding $f(x-ct)$ and $g(x+ct)$ for the two different cases. For $x < ct$ we have:

$$u(x,t) = -\frac{c}{8\pi}\sin\left(\frac{4\pi(x-ct)}{c}\right) + \frac{1}{2c}e^{x-ct} - \frac{1}{2c}e^{-x-ct}$$

and for $x > ct$:

$$u(x,t) = +\frac{1}{2c}e^{-x+ct} - \frac{1}{2c}e^{-x-ct}$$

The solution for $ct = 0$, $ct = 1$ and $ct = 10$ is plotted below



Note that in the first graph the solution is zero everywhere.

6 The question leads you through the solution to this. If not, look at the relevent page of Kreyszig.