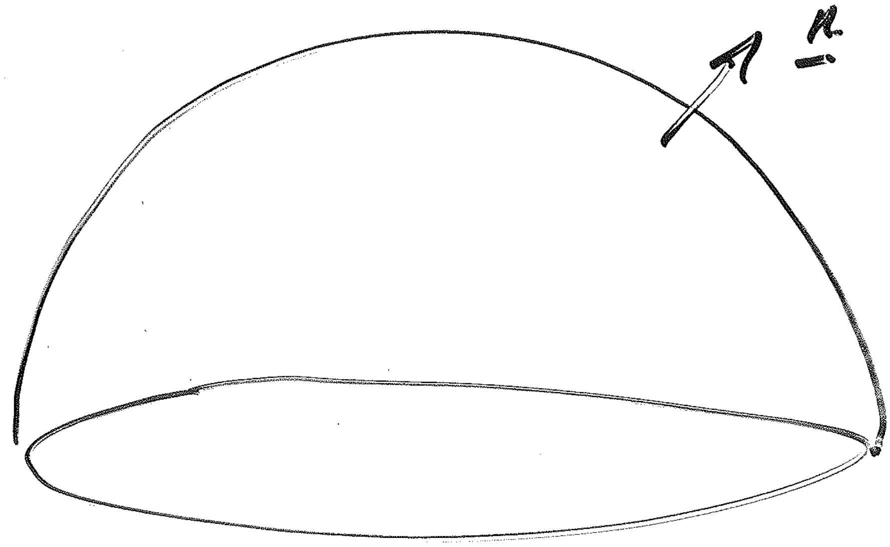
Q7 (a) State Gauss's Divergence Theorem.

(2 marks)

(b) Given $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$, and the region bounded by the hemisphere, $z = \sqrt{1 - x^2 - y^2}$ and $z = 0$, use Gauss's Divergence Theorem to determine the surface integral of $\mathbf{F}$ taken over the curved surface.

(8 marks)



$S: \quad S(x, y, z) = 0.$

$\underline{n}$ to S ‖'l ∇S.

unit normal, $\dfrac{\nabla S}{|\nabla S|}$ $\quad d\underline{S}$

$$\overset{\underline{\underline{I}}}{\oint\!\!\oint_S \underline{F}\cdot\hat{\underline{n}}\, dS} = \overset{\underline{\underline{II}}}{\iiint \operatorname{div} \underline{F}\, dV}$$

(whole surface) (enclosed volume)

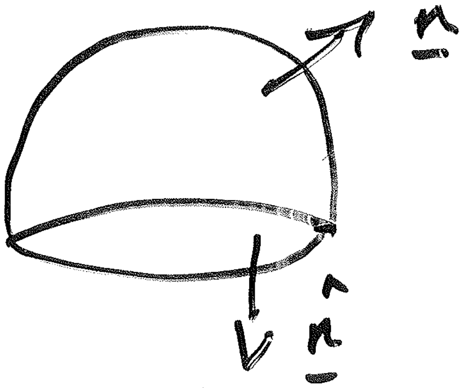
Integrals are <u>scalar</u>.

$\underline{\underline{II}}: \operatorname{div} \underline{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$

$= 0 + 0 + 0 = 0.$

so: $\underline{\underline{II}} = 0.$

I :



$$\oiint \underline{F}\cdot\hat{\underline{n}}\,dS = 0.$$

So curved part integral

$= -$ base integral.

On base: $\hat{\underline{n}} = -\underline{k}$.

$$\text{Curved} \int = \iint_{\substack{x^2+y^2\le 1\\(z=0)}} \underline{F}\cdot\underline{k}\;dx\,dy$$

$$= \iint_{x^2+y^2\le 1} x^2\,dx\,dy.$$

Use polars $x = r\cos\theta$, $y = r\sin\theta$.

$0 \le r < 1$, $0 \le \theta < 2\pi$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r, \quad \int_0^{2\pi}\!\!\int_0^1 r^2\cos^2\theta\; r\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\,d\theta \cdot \int_0^1 r^3\,dr = \pi/4.$$

Q8 The vector field $\mathbf{F} = (x^2 - y)\mathbf{i} + (y^3 + 2x)\mathbf{j}$ is not conservative because curl $\mathbf{F} = 3\mathbf{k}$. Establish that $\mathbf{F}$ is not conservative by evaluating the line integral of $\mathbf{F}$ taken from $(0,0)$ to $(1,1)$ along the paths given in (a) and (b) below.

(a) Along the straight line $y = x$.

(5 marks)

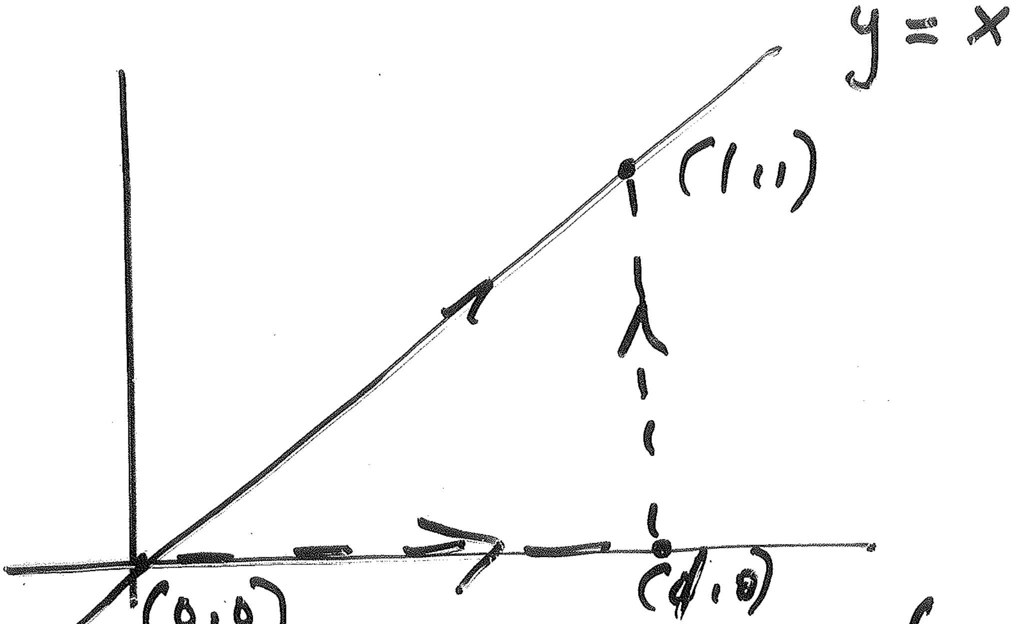
(b) Along the straight line paths $(0,0)$ to $(1,0)$, then to $(1,1)$.

(5 marks)

Not conservative:

$$\cancel{\mathrm{Curl}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ & & 0 \\ & & 0 \end{vmatrix}}$$

$$\mathrm{Curl}\,\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y & y^3+2x & 0 \end{vmatrix}$$



On $y=x$, take $x=y=t$, parameter.
so $dx = dy = dt$.

$$\int: \int_0^1 \{(t^2-t)\,dt + (t^3+2t)\}\,dt = $$

$$1/3 - 1 + 1/4 + 1.$$
$$= 7/12.$$

On OX: $y = 0$, $dy = 0$.

$$\int \underline{F} \cdot d\underline{r} = \int_0^1 x^2\,dx + 0 = 1/3.$$

On OY: $x = 1$, $dx = 0$

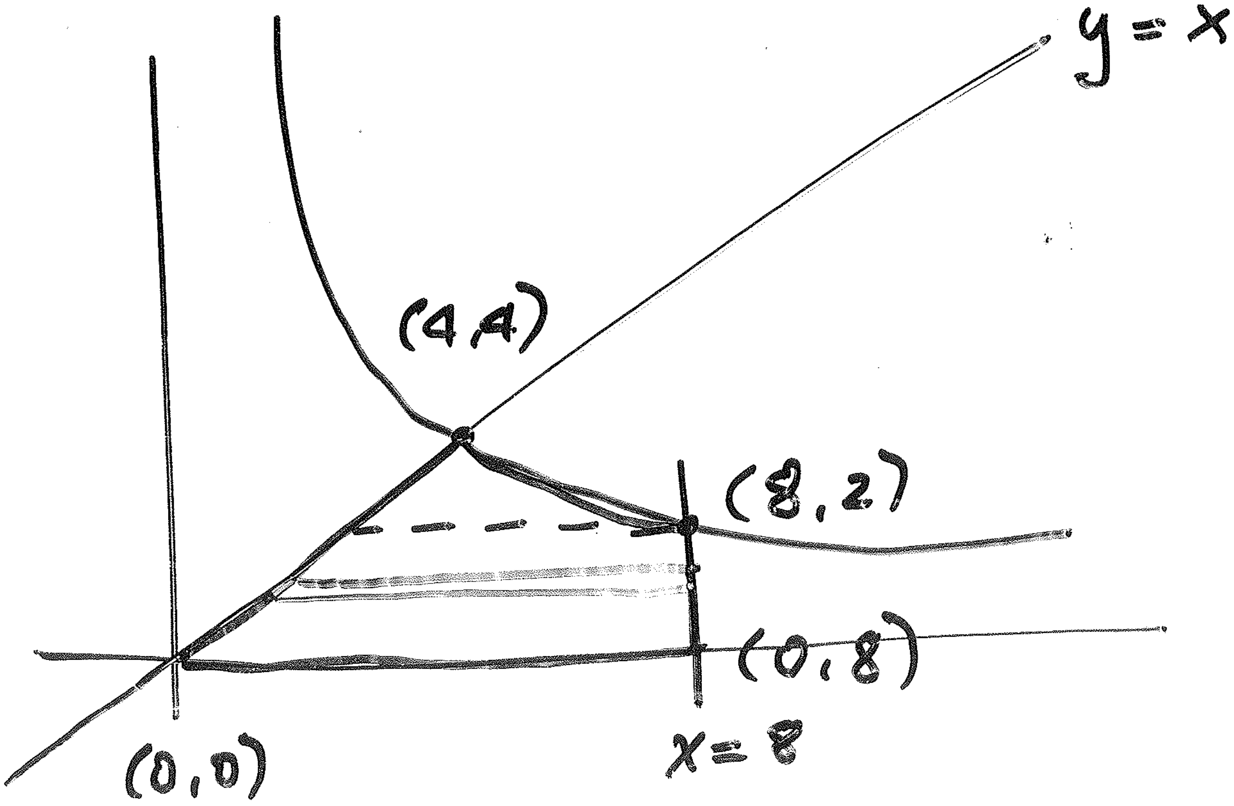
$$\int \underline{F} \cdot d\underline{r} = \int_0^1 \overset{(y^3+2)}{\cancel{y^3}}\,dy = 2\tfrac{1}{4} = 9/4.$$

Q9 (a) Sketch the region A bounded by $y = x$, $xy = 16$, $y = 0$ and $x = 8$.

(3 marks)

(b) By taking the x integration first, evaluate $\int x^2 dA$ over the region.

(7 marks)



$\int x^2\, dA$.

$$\overset{\mathrm{I}}{\int_0^2\int_y^8 x^2\, dx\, dy} + \overset{\mathrm{II}}{\int_2^4\int_y^{16/y} x^2\, dx\, dy}$$

$\underline{\mathrm{I}}$: $\displaystyle\int_0^2 \left[x^3/3\right]_y^8 dy = \int_0^2 \left(8^3/3 - y^3/3\right) dy$

$= \cancel{\tfrac{1}{3}\left(8^3\cdot y - y^4/12\right)}$

$8^3/3 - y^3/3 = \frac{1}{3}\left(8^3 - y^3\right)$.

$\boxed{= 340}$

Integrate wrt y,

$\displaystyle\frac{1}{3}\int_0^2 \left(8^3 - y^3\right) dy = \frac{1}{3}\left(8^3\cdot 2 - 2^4/4\right)$

$= (1024 - 4)/3.$