Polynomial Interpolation

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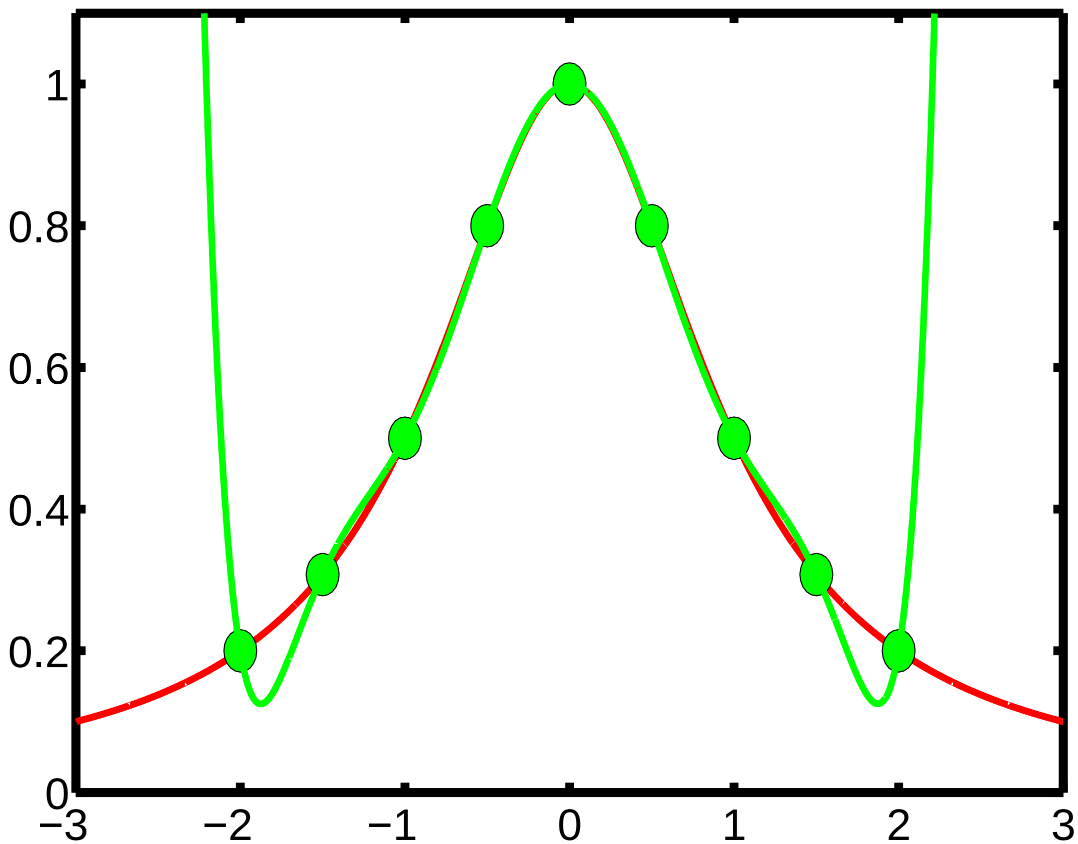
Motivation

- Polynomials $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0$
  - easy to evaluate
  - easy to differentiate
  - easy to integrate
- Useful to approximate other functions by polynomials as basis for:
  - approximate integration
  - numerical solution of differential equations
- Also use polynomials to put a *smooth curve* through experimental data points

Basic idea of interpolation

- Take n points on a curve $y = f(x)$ and find $p(x)$ of degree $n-1$ which goes through them
- Here $f(x) = 1/(1+x^2)$



- $n = 9$ data points
- $p(x)$ degree eight
- higher and higher degrees give better approx (in some sense)

Simplest case: linear interpolation

- Given two points (x_1, y_1) and (x_2, y_2) find the straight line $(y = mx + c)$ through them

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$$

- OR: suppose straight line is $y = a_1 x + a_0$.
Formulate linear system to be solved for a_0 and a_1

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

  - and apply Gaussian elimination etc. etc.

Quadratic interpolation

- Try to interpolate three data points $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$ with quadratic $y = a_2 x^2 + a_1 x + a_0$
- Same idea as linear case: solve linear system for a_0, a_1, a_2

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

  - and apply Gaussian elimination etc. etc.

Example

- Find the interpolating quadratic $y = p_2(x)$ for the three (x, y) pairs $(0, 1)$, $(1, 2)$, and $(2, 7)$.

General theory of interpolation

- Same linear systems idea works for higher and higher degree polynomials
- How do we know left hand matrix is non-singular?
- Theorem:

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, with $x_1 < x_2 < \ldots < x_n$, there exists a unique interpolating polynomial

$$y = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0,$$

with degree less than or equal to $(n-1)$.

Lagrange interpolating polynomials

(alternative method for doing interpolation)

- Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, define

$$L_i(x) = \frac{\Pi_{j \neq i}(x - x_j)}{\Pi_{j \neq i}(x_i - x_j)}$$

- so that

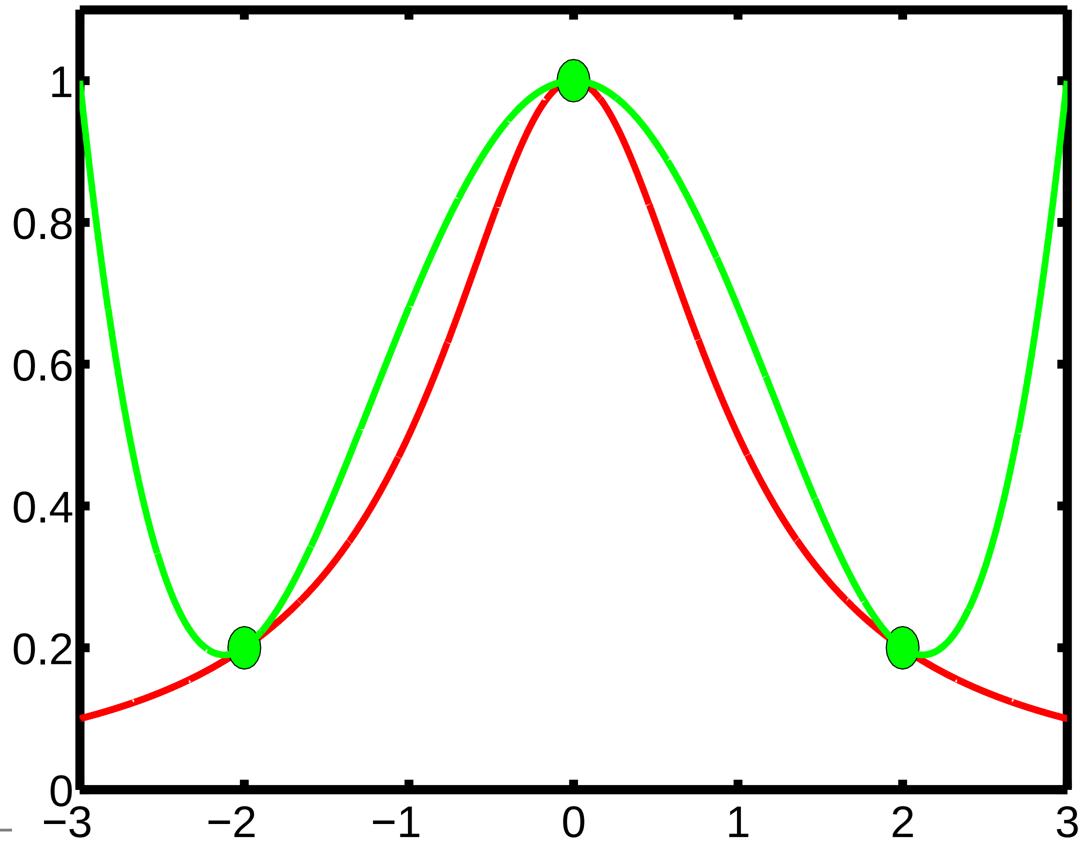
$$L_i(x_j) = \begin{cases} 0 & j \neq i, \\ 1 & j = i. \end{cases}$$

- Interpolating polynomial $p_{n-1}(x)$ of degree $n-1$ is

$$p_{n-1}(x) = y_1 L_1(x) + y_2 L_2(x) + \ldots + y_n L_n(x)$$

Hermite interpolation (I)

- Given n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ on a curve, find interpolating polynomial which also matches gradient of curve at given points



- $n = 3$ data points
- six constraints
- $p(x)$ degree five (quintic)
- in general degree $2n - 1$ is needed

Hermite interpolation (II)

(example of how to construct)

- Find cubic poly $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ which passes through (x_1, y_1) and (x_2, y_2) with given derivative values y_1', $y_2\prime$
- $p'(x) = 3a_3x^2 + 2a_2x + a_1$
- Formulate linear system

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 1 & 2x_2 & 3x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{pmatrix}$$