Engineering Mathematics 2: Numerical Analysis
Exercise Sheet 1

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Review of root finding

1. For each of the following functions and intervals, state whether the Intermediate Value Theorem applies, and draw conclusions concerning the number of roots in the given interval:

(a)	$\sinh(x+4)=0$	$(-5,0)$
(b)	$x^4-9x^2+20=0$	$(-1,-2.2)$
(c)	$x^4-9x^2+20=0$	$(1,3)$
(d)	$(x^3+2x^2+2)/(4-x)=0$	$(3,5)$
(e)	$(x^3+2x^2+2)/(4-x)=0$	$(-1,-3)$
(f)	$\ln(\lvert x\rvert)=0$	$(-0.5,1.1)$

2. Use the Intermediate Value Theorem to show that the equation

$$3x^3-5x^2-4x+4=0$$

has a root in the interval $(0,1)$. Use the Interval Bisection Method to obtain an interval of length less than $1/8$ containing the solution. How many iterations would be needed to obtain an approximate solution with error less than 10^{-6}? Is there another method of finding this root?

3. Show that the equation $\exp(x)-3x-1=0$ has a root in the interval $(1,3)$. How might you prove that this is the unique positive root?

4. Find a re-arrangement of the equation $\exp(x)-3x-1=0$ which will converge to the unique positive root when the Fixed Point Iteration is applied.

5. (a) Carefully sketch graphs of the functions $y=\tan(x)$ and $y=x$ for $x\in(0,3\pi)$. Hence determine appropriate intervals in which to seek the *first and second* positive solutions to the equation $x=\tan(x)$.

   (b) Use the theoretical formula for the convergence rate to explain why the Fixed Point Iteration will fail to converge to these roots.

6. (a) The cubic $2x^3+3x^2-3x-5=0$ has a root near $x=1.25$. Show that this equation can be rearranged into any of the following three forms suitable for fixed-point iteration:

i. $x = ((5 + 3x - 3x^2)/2)^{1/3}$

ii. $x = ((5 + 3x)/(2x + 3))^{1/2}$

iii. $x = (2x^3 + 3x^2 - 5)/3$.

(b) Use fixed-point iteration on the rearranged equation (i) with an initial guess of $x_0 = 1.2$ in order to find the root to 4 decimal places. Use the ratio of successive differences to estimate the rate of convergence. Compare this estimate with the rate of convergence calculated theoretically using the formula in your notes. Do they agree?

(c) Repeat part (b) for the rearrangement (ii) using $x_0 = 1.2$. Which method converges fastest? Why?

(d) Try a few iterations using rearrangement (iii). What goes wrong? Use the theoretical formula for the rate of convergence to explain why.

7. (a) Show that there is a zero of the polynomial $x^3 - x^2 - x - 1$ in the interval $(0, 2)$.

(b) Use the bisection method to locate a root in this interval to within 1 decimal place.

(c) Use a couple of iterations of Newton's method to refine your approximation to the root.

8. (a) The function $f(x) = (x - 1)^3(x - 2)$ has roots at $x = 1$ and $x = 2$. Using starting values $x_0 = 1.1$ and $x_0 = 2.1$, find the number of iterations it takes to locate each of the roots to 3 decimal places using Newton's method. Is the convergence quadratic in both cases? Use the error analysis of Newton's method to explain the convergence rates.

(b) What happens if $x_0 = 1.75$ is used as a starting value for Newton's method? Explain with the aid of a sketch graph.

9. * The Fixed Point Iteration method can be defined for systems of equations in much the same way as it can for scalar equations. One re-arranges the system into the form

$$\mathbf{x} = \mathbf{g}(\mathbf{x}),$$

and then performs the iteration

$$\mathbf{x}_{n+1} = \mathbf{g}(\mathbf{x}_n).$$

Apply several steps of this method to find an approximate solution of the system

$$x - \cos y = 0,$$
$$y - \cos x = 0,$$

starting with the initial point $(x_0, y_0) = (0, 0)$. Comment on your results. By considering the scalar equation $x - \cos(\cos x) = 0$, find the number of roots of the system.