Which Fourier formula?

In general, for some general function f with period T we have

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

where

$$\omega = \frac{2\pi}{T}$$

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt \\ a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt \end{aligned} \tag{1}$$

Even functions

- When f is even, if you do the integrals in 1 you will find that the sine terms b_n are all 0. This is in the notes LSPDE1.
- When f is even, $f(t)\cos(n\omega t)$ is also even (EVEN x EVEN = EVEN).
- So
$$\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt = 2\frac{2}{T}\int_{0}^{T/2} f(t)\cos(n\omega t)dt$$
- and if we let $L = T/2$ we get the half-range formula
$$a_n = \frac{2}{L}\int_0^L f(t)\cos(\frac{n\pi t}{L})dt.$$

Odd functions

- When f is odd, if you do the integrals in 1 you will find that the cosine terms a_n are all 0. This is in the notes LSPDE1.
- When f is odd, $f(t)\sin(n\omega t)$ is even (ODD x ODD = EVEN).
- So

$$\frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt = 2\frac{2}{T}\int_{0}^{T/2} f(t)\sin(n\omega t)dt$$

- and if we let $L = T/2$ we get the half-range formula

$$b_n = \frac{2}{L}\int_{0}^{L} f(t)\sin(\frac{n\pi t}{L})dt.$$

So far these have all been the *same* formula, just special cases.

But these special cases are used in a couple of ways.

(1) Periodic extensions

If you have a function $f(t)$ defined on an interval $[0, L]$, and not defined anywhere else, then you could make an odd function $f_o(t)$ out of it by forming it's *odd periodic extension*. You could make an even function out of it by forming the *even periodic extension* $f_e(t)$. Because f_o is odd, its Fourier series expansion is given by the odd version of the general formula. Because f_e is even, it's Fourier series expansion is given by the *even* reduced version of the general formula. But both use the same general formula, it's just that some of terms are 0 and if we want to, we can label $T/2 = L$.

A key point is that both the extensions lead to a Fourier series representation for $f(t)$ on the interval $[0, L]$. They will differ outside that interval.

(2) PDE's

Here's where it all comes together. When we solve a PDE using separation of variables, we often get something like

$$u(x,t) = X(x)T(t) = \quad \text{(sum of terms like)} \;\; b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right),$$

and we need to match an initial condition $u(x,0) = f(x)$.

So here, because of the structure of the PDE, we *have* to get a series for $f(x)$ in terms of sines only, because when $t = 0$ the cosine term is 1 and we must match

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Fortunately, we have two ways to get a Fourier series representation for $f(x)$ on the interval $[0, L]$: the sine way (odd) and the cosine way (even). In this case, we want sine functions in the series for $f(x)$, so we would solve

$$b_n = \frac{2}{L}\int_0^L f(t)\sin(\frac{n\pi t}{L})dt.$$