5. The separation of variables method

A 'try it and see' technique. Satisfying the boundary conditions. Solution process for wave, heat, and Laplace equations (all but last step). But how to find the unknown coefficients of the periodic functions? Use Fourier series! More worked examples. Linear superposition principle. Inhomogeneous boundary conditions or equations.

Outline of the method

1. **Separate the variables**
   Assume, for example, that
   $$u(x,t) = X(x)T(t).$$
   Substitute this into the PDE to get 2 ODE's for X and T *separately.*
2. **Decide on the sign of the separation constant**
   The constant arises when you separate the variables. More on this later.
3. **Solve the separated ODE's**
   You get, for example, ODE's to solve for $X(x)$ and $T(t)$ that depend on the constant in Step 2.
4. **Solve the (homogeneous) boundary conditions**, so that you know what $X(t)$ and $T(t)$ are, and reconstruct the funtion, for example $u(x,t)$ that you need, using $u(x,t) = X(x)T(t)$.
5. **Check** that the $u(x,t)$ that you have actually solves the problem.

5.1 Solving the wave equation

Consider the wave equation on a finite domain

$$u_{tt} = c^2 u_{xx}, \qquad 0 \leq x \leq L, \quad t \geq 0,$$

subject to homogeneous boundary conditions and a simple initial condition.

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0$$

for some given function $f(t)$.

Step 1 The basic idea is to TRY to find a solution that is a function of x times a function of t. That is, we write

$$u(x,t) = X(x)T(t),$$

Substituting this form into the PDE we get

$$X(x)T''(t) = c^2 X''(x)T(t)$$

which gives

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} \tag{5.1}$$

Now, the left-hand side of (5.1) is a function of time t, while the right-hand side is a function of space x. The only way that this can be true for all x and t is if both functions are actually equal to a constant. Hence

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = \text{const.} \tag{5.2}$$

This constant is called the **separation constant**. The question remains what sign this constant should have. We proceed by trial and error to see what fits the boundary and initial conditions:

Step 2 first guess Try first a *positive* constant. Hence we write (5.2)

$$\frac{T''(t)}{T(t)} = c^2\frac{X''(x)}{X(x)} = k^2 > 0$$

Then we get two separate linear ODEs to solve:

$$\begin{aligned} T''(t) &= k^2T(t) \qquad (5.3)\\ X''(x) &= (k/c)^2X(x) \qquad (5.4) \end{aligned}$$

Now, suppose we solve the T equation (5.3) first, we then get

$$T(t) = Ae^{-kT} + Be^{kT}$$

for arbitrary constants A and B. Satisfying the initial condition $u_t(x, 0) = 0$ we get

$$X(x)(-kA + kB) = 0, \qquad \text{hence } A = B \text{ unless } X = 0 \text{ everywhere}$$

But if $A = B$ the the solution $T(t)$ tends to $+\infty$ as $t \to \infty$. This is not wave-like motion!

Step 2 again Hence we should take the original separation constant to be *negative*. That is we write (5.2) in the form

$$\frac{T''(t)}{T(t)} = c^2\frac{X''(x)}{X(x)} = -k^2 < 0$$

Step 3 Thus we get the two separate linear ODEs:

$$\begin{aligned} T''(t) &= -k^2T(t) \qquad (5.5)\\ X''(x) &= -(k/c)^2X(x) \qquad (5.6) \end{aligned}$$

Now, solving the T equation (5.5) we find

$$T(t) = A\cos(kt) + B\sin(kt)$$

which *is* wave-like!

Now, from the homogeneous initial condition $u_t(x,0) = 0$ we get that

$$0 = X(x)(-kA\sin(0) + Bk\cos(0)) = X(x)Bk$$

which gives us $B = 0$. Hence $T(t) = A\cos(kt)$ for some arbitrary constant A. (Note we also haven't solved for k yet!).

Next, we solve the X equation (5.6) which gives

$$X(x) = C\cos((k/c)x) + D\sin((k/c)x)$$

for arbitrary constants C and D.

Step 4

Remember we had these boundary conditions and a simple initial condition:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0$$

Now we need to solve the boundary conditions

$$\begin{aligned} 0 = u(0,t) &= (C\cos(0) + D\sin(0))T(t) = CT(t), \qquad (5.7) \\ 0 = u(L,t) &= (C\cos(kL/c) + D\sin(kL/c))T(t). \qquad (5.8) \end{aligned}$$

From (5.7) we get $C = 0$, hence from (5.8) we have

$$D\sin(kL/c) = 0, \qquad \text{hence } ckL = n\pi.$$

This gives us the condition for k, namely

$$\frac{k}{c} = \frac{n\pi}{L}, \qquad \text{for some integer } n$$

Hence we have the solution

$$u(x,t) = X(x)T(t) = b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right),$$

where $b_n = AD$, and we still need to decide which value of n to take.

At this stage we should check that we satisfy the PDE and the boundary + initial conditions. We have got a function that meets

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u_t(x,0) = 0 \tag{5.9}$$

but NOT $u(x,0) = f(x)$.

So how do we do it?

Well, we've still got this b_n and these n's in our $u(x,t)$ ... and all 3 requirements that we have met have got 0's in them. So we have some flexibility left to meet $u(x,0) = f(x)$.

Let

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right)$$

ie the solution that we found.

KEY POINT: If $u_2(x,t)$ and $u_3(x,t)$ meet the conditions (5.9), then so does $u_2(x,t) + u_3(x,t)$.

We can use this to satisfy the initial condition $u(x,0) = f(x)$ by finding a special sum of the u_n's that happens to be just the right function ($f(x)$) at $t = 0$. We would need to find the right b_n's such that

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \tag{5.10}$$

If we could find a set of b_n to solve this equation. Then we would be done!

But ... we already know how to do this ... using Fourier series!

So, the general solution to this PDE satisfying the boundary and initial conditions is:

$$u(x,t)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{nc\pi t}{L}\right),$$

where the b_n's solve the equation (5.17), ie are the Fourier coefficients of $f(x)$.

Subtle changes in the boundary conditions lead to different forms of solution. For example, we can replace these boundary conditions with simply supported rather than pinned ends. Or, rather than an initial profile $u(x,0)=f(x)$, we can also specify an initial velocity $g(x)$ at every point along the string:

Worked example 5.1 *Solve the wave equation*

$$u_{tt}=c^2u_{xx}, \qquad 0\le x\le L, \quad t\ge 0,$$

subject to homogeneous boundary conditions and an inhomogeneous initial condition:

$$u_x(0,t)=0, \quad u_x(L,t)=0, \quad u(x,0)=0, \quad u_t(x,0)=g(x)$$

for some given function $g(x)$.

Alternatively, we could choose the initial conditions to be homogeneous, and make one of the boundary conditions inhomogeneous (see example sheet).

5.2 Solving the heat equation

Worked example 5.2 *Show that the general solution to the heat equation*

$$u_t=\alpha^2u_{xx}, \qquad 0\le x\le L, \qquad t>0,$$

subject to boundary conditions that hold the ends fixed at zero temperature

$$u(0,t) = 0, \qquad u(L,t) = 0$$

and an initial condition

$$u(x,0) = h(x)$$

for some given function $h(x)$*.*
Preview: the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\alpha^2\pi}{L}\right)t},$$

where the b_n*'s solve the Fourier series problem*

$$h(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) .$$

Answer

Step 1 The basic idea is to TRY to find a solution that is a function of x times a function of t. That is, we write

$$u(x,t) = X(x)T(t),$$

Substituting this form into the PDE we get

$$X(x)T'(t) = \alpha^2 X''(x)T(t),$$

which simplifies to

$$\frac{T'(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)},$$

Now, the left-hand side of (5.11) is a function of time t, while the right-hand side is a function of space x. The only way that this can be true for all x and t is if both functions are actually equal to a constant. Hence

$$\frac{T'(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)} = \text{const.} \tag{5.11}$$

This constant is called the **separation constant**. The question remains what sign this constant should have. We proceed by trial and error to see what fits the boundary and initial conditions:

Step 2 Try first a *positive* constant. Hence we write (5.11) as:

$$\frac{T'(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)} = k^2 > 0$$

Then we get two separate linear ODEs to solve:

$$\begin{aligned} T'(t) &= k^2 T(t) \qquad (5.12) \\ X''(x) &= (k/\alpha)^2 X(x) \end{aligned}$$

Now, suppose we solve the T equation (5.12) first, we then get

$$T(t) = A \exp kT,$$

for an arbitrary constant A. Now this solution tends to $+\infty$ as $t \to \infty$. This is not a diffusion-like process (heat decays, not blows up!)

Hence we should take the original separation constant to be *negative*. That is we write (5.2) in the form

$$\frac{T'(t)}{T(t)} = \alpha^2 \frac{X''(x)}{X(x)} = -k^2 < 0$$

Thus we get the two separate linear ODEs:

$$T'(t) = -k^2 T(t) \qquad (5.13)$$

$$X''(x) = -(k/\alpha)^2 X(x) \qquad (5.14)$$

Step 3 Now, solving the T equation (5.13) we find

$$T(t) = Ae^{-k^2 t},$$

for some arbitrary constant A, which does decay.

Next, we solve the X equation (5.14), which is the equation for simple harmonic motion and has the general solution

$$X(x) = B\cos((k/\alpha)x) + C\sin((k/\alpha)x)$$

for arbitrary constants B and C.

So we have found the general solution $u(x,t) = X(x)T(t)$, up to arbitrary constants A, B, C and k.

Step 4 To find these constants, we need to satisfy the boundary and initial conditions. We start by taking the boundary conditions

$$0 = u(0,t) = (B\cos(0) + C\sin(0))T(t) = BT(t), \quad (5.15)$$

$$0 = u(L,t) = (0\cos(kL/\alpha) + C\sin(kL/\alpha))T(t). \quad (5.16)$$

From (5.15) we get $B = 0$, hence from (5.16) we have

$$C\sin(kL/\alpha) = 0, \qquad \text{hence } \tfrac{kL}{\alpha} = n\pi.$$

This gives us the condition for k, namely

$$\frac{k}{\alpha} = \frac{n\pi}{L}, \qquad \text{for some integer } n$$

Hence we have the solution

$$u(x,t) = X(x)T(t) = b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\alpha\pi}{L})^2 t},$$

where $b_n = AC$, and we still need to decide which value of n to take.

Step 5 At this stage we should check that we satisfy the PDE and the boundary + initial conditions. So, we still have to satisfy the initial condition $u(x,0) = h(x)$.

Step 6 Allowing for a possible sum of different n's we can formally write

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Note, that this is just the Fourier half-range sine series expansion for the function $h(x)$. Hence we know that

$$b_n = \frac{2}{L} \int_0^L h(x) \sin\left(\frac{n\pi t}{L}\right) dx \tag{5.17}$$

So, the general solution to this PDE satisfying the boundary and initial conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\alpha\pi}{L}\right)t},$$

where the b_n's solve the equation (5.17).

Subtle changes in the boundary conditions lead to different forms of solution. For example, we can replace these boundary conditions with conditions that the bar is *insulated* at each end. That is, there is no heat flux:

$$\frac{\partial}{\partial x} u(0,t) = 0, \qquad \frac{\partial}{\partial x} u(L,t) = 0,$$

Such boundary conditions can be shown to lead to a similar general solution to the heat equation, but with *cosine* rather than sine terms:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{a_0}{2} + a_n \cos\left(\frac{n\pi x}{L}\right)\right] e^{-\left(\frac{n\alpha\pi}{L}\right)^2 t},$$

where the $a_n's$ are the Fourier half-range cosine coefficients of $h(x)$.

5.3 Solving Laplace's equation

Since Laplace's equation involves only spatial co-ordinates, (x, y) (or (x, y, z) in three dimensions), it is quite natural to pose Laplace's equation on domains of any shape. E.g. circular domains (find the shape of a drumskin) or complex curvy shapes (find the incompressible, irrotational flow through a curved river bed).

However, in these lectures we shall concentrate only on the simplest case of a rectangular domain in 2D.

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \qquad 0 < y < b. \tag{5.18}$$

On each boundary ($x = 0$ or a, $y = 0$ or a) there are two types of homogeneous boundary condition we can pose, that either the solution is zero on the boundary (DIRICHLET B.C.'s):

$$u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, b) = 0; \tag{5.19}$$

or its **normal** derivative is zero on the boundary (NEUMANN B.C.'s):

$$u_x(0, y) = 0, \quad u_x(a, y) = 0, \quad u_y(x, 0) = 0, \quad u_y(x, b) = 0. \tag{5.20}$$

A mixture of Dirichlet on some parts of the boundary and Neumann on others is also possible.

Note that we usually need some kind of inhomogeneous boundary condition in order to get a non-trivial solution.

Worked example 5.3 *Solve the Laplace equation (5.18) subject to uniform:*

(a) *Dirichlet boundary conditions (5.19);*

(b) *Neumann boundary conditions (5.20).*

The method of separation of variables is especially well suited to the case where we have **one** inhomogeneous boundary condition. e.g.

Example: *Solve the Laplace equation on a rectangular domain (5.18) subject to the inhomogeneous Neumann boundary conditions*

$$u_x(0, y) = u_x(a, y) = 0, \quad u_y(x, 0) = 0, \quad u_y(x, b) = f(x)$$

for some given function $f(x)$.

Such a problem could describe, for example, the electrostatic potential $u(x, y)$ in a rectangular device whose boundaries on three sides are insulated (electromagnetically shielded) but is held at a constant potential $f(x)$ on the boundary $\{y = b\}$.

To solve this problem we use the method of separation of variables:

Step 1: separate the variables

$$u(x, y) = X(x)Y(y)$$

Substituting this into the PDE gives us

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

which simplifies to

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const.}$$

Step 2: decide on sign of separation constant

Note that if we choose const. $= k^2 > 0$ then we get exponential solutions for $X(x)$ and sinusoidal solutions for $Y(y)$.
Alternatively, const. $= -k^2 < 0$ then we get exponential solutions for $Y(y)$ and sinusoidal solutions for $X(x)$.

You could just try both and see which works (this is a perfectly valid approach in a 'trial and error' method). Alternatively, you could appeal to the guiding principle that when we come to pose the inhomogeneous boundary condition $u_y(x, b) = f(x)$ we are going to be looking for a function of x, and we want to end up by expressing this function of x as a sum of sines or cosines.

Hence we choose const. $= -k^2 < 0$.

Step 3: solve the separated ODEs

For the X equation we have

$$X''(x) = -k^2 X(x),$$

which gives

$$X(x) = A \sin kx + B \cos kx,$$

for arbitrary constants A and B. For the Y equation we get

$$Y(y) = \tilde{C} e^{-ky} + \tilde{D} e^{ky},$$

for arbitrary constants C and D. However it is useful to express this another way using the fact that

$$\cosh(z) = \frac{1}{2}(e^z + e^{-z}) \qquad \sinh(z) = \frac{1}{2}(e^z - e^{-z})$$

(we do this for convenience because cosh is an even function and sinh is an odd function). So we get

$$Y(y) = C \cosh(ky) + D \sinh(ky)$$

Hence, so far we have

$$u(x,t) = X(x)Y(y) = (A \sin kx + B \cos kx)(C \cosh(ky) + D \sinh(ky)).$$

Step 4: solve the homogeneous boundary conditions

Let us first pose the boundary conditions at $x = 0$ and $x = a$:

$$\begin{aligned} u_x(0,y) &= (Ak \cos 0 - Bk \sin 0)Y(y) = AkY(y) = 0 \\ u_x(a,y) &= (Ak \cos ak - Bk \sin ak)Y(y) = 0. \end{aligned}$$

Note that both equations must hold *for all* values of y. Hence, the first equation gives us $A = 0$, and the second that $B \sin ak = 0$, hence $ak = n\pi$ for some integer n.

Now, consider the boundary condition at $y = 0$:

$$u_y(x,0) = X(x)(Ck \sinh(0) + Dk \cosh(0)) = X(x)Dk,$$

which must be true for all x. Hence $D = 0$. So, letting $AC = A_n$, we have

$$u(x,y) = X(x)Y(y) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a} \cosh\frac{n\pi y}{a}$$

Step 5: check that what you have solves the PDE and B.Cs(!)

Step 6: pose the inhomogeneous boundary condition

$$f(x) = u_y(x,b) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}$$

Hence if we let

$$A_n = \frac{1}{\sinh\frac{n\pi b}{a}} a_n,$$

then we'd have

$$u_y(x,b) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{a}$$

so to meet the boundary condition we have to use the a_n's that satisfy

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{a}$$

ie the Fourier half-range cosine series for $f(x)$.

Step 7: solve the inhomogeneous boundary condition

Solved - as soon as we find the right a_n's.

5.4 Solving the final boundary condition

Now, the half-range Fourier series are ideal for solving the final step of the separation of variables method. For example, if we have *Dirichlet* boundary conditions (solution $u = 0$ on the boundary) then we find that sines satisfy this — we could imagine that the solution extends beyond the boundary in an ODD fashion.

Similarly if we have *Neumann boundary conditions* (derivative $u_x = 0$ on the boundary) we find that cosines satisfy this — we could imagine reflecting the solution in the boundary in an EVEN fashion.

So the final step of the separation of variables method:
Step 7 - solving the inhomogeneous boundary condition reduces to finding a Fourier half-range cosine or sine expansion.

We can now complete the two extended examples that we treated in sections 5.1 and 5.3

Worked example 5.4 *Solve the wave equation on the finite domain*

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le L, \quad t \ge 0,$$

subject to homogeneous boundary conditions and a simple initial condition.

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = 0$$

for the specific case $L = 4$ *and*

$$f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4 - x, & 2 < x \le 4 \end{cases} \tag{5.21}$$

Worked example 5.5 *Solve the Laplace equation on a rectangular domain subject to the inhomogeneous Neumann boundary conditions*

$$u_x(0,y) = u_x(a,y) = 0, \quad u_y(x,0) = 0, \quad u_y(x,b) = f(x)$$

for the particular case $a = 4$ *and* $b = 2$ *and* $f(x)$ *is the same function as in Worked example 4.1*

5.5 Solving inhomogeneous equations

When solving linear ODEs one of the most important properties is that of linear superposition. This is also true for linear PDEs:

Theorem: Principle of linear superposition *Consider a general linear PDE which we write in the form*

$$L\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) \tag{5.22}$$

where $\boldsymbol{u}$ *is a vector of dependent variables,* $\boldsymbol{x}$ *is a vector of independent variables and* L *is a linear operator – which may include linear partial derivative terms. Then if* $\boldsymbol{v}(\boldsymbol{x})$ *is* **any** *solution that satisfies*

$$L\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})$$

then we may write the general solution of (5.22) in the form

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \quad \textit{where } w \textit{ solves the homogeneous equation} \quad L\boldsymbol{w} = 0.$$

Note that this theorem tells us nothing about boundary conditions. In general the solution of the homogeneous problem will solve different boundary conditions.

To illustrate how this works, it is best to proceed by example, where we restrict to the case that the inhomogeneous term is a pure function of just *one* of the independent variables. Then the step of finding the solution v is just exactly the same as finding *particular integrals* of linear ODEs.

Worked example 5.6 *Consider the Poisson equation*

$$u_{xx} + u_{yy} = \sinh(x), \qquad 0 \le x < 1, \quad 0 \le y \le 1$$

subject to homogeneous Dirichlet boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0.$$

Find a solution $w(x)$ to the inhomogeneous problem such that the solution $u = w + v$ where v solves the Laplace equation. Find the new boundary conditions satisfied by $v(x,t)$.

Worked example 5.7 *Solve the equation for the diffusion of heat in the presence of a spatially uniform source term*

$$u_t - \alpha^2 u_{xx} + e^{-\kappa^2 t}, \qquad 0 \le x \le L, \qquad t > 0,$$

subject to boundary conditions that hold the ends fixed at zero temperature

$$u(0,t) = 0, \qquad u(L,t) = 0$$

and an initial condition

$$u_t(x,0) = 0,$$

by expressing the solution as $u = v + w$ where w solves the homogeneous heat equation. Find the new boundary and initial conditions satisfied by w.

5.6 Inhomogeneous boundary conditions

The principle of linear superposition also applies to problems with tricky boundary conditions.

For example, suppose we have the heat equation

$$u_t = \alpha^2 u_{xx}, \qquad 0 \le x \le L, \quad 0 \le t < \infty,$$

with boundary condition

$$u(0,t) = c_1, \quad u(L,t) = c_2$$

for constants c_1 and c_2, and some initial condition

$$u(x,0) = h(x) \, .$$

Then, if we can find a solution to the $v(x,t)$ to the PDE that satisfies

$$v(0,t) = c_1, \quad v(L,t) = c_2, \quad v(x,0) = 0,$$

then if we write $u = v + w$, then w solves the *same* PDE but with new boundary conditions satisfied by w are

$$w(0,t) = u(0,t) - v(0,t) = 0, \quad w(L,t) = u(L,t) - v(L,t) = 0 \, .$$

Hence we have reduced the problem to one we know how to solve with *homogeneous* boundary conditions + the same initial condition

$$w(0,t) = 0, \quad w(L,t) = 0, \quad w(x,0) = h(x).$$

But how to find such a solution v? In this case its easy since we note that the function

$$v(x) = c_1 + \frac{c_1 - c_2}{L} x$$

Worked example 5.8 *Consider the wave equation on a finite domain*

$$u_{tt} = c^2 u_{xx}, \qquad 0 \leq x \leq 2, \quad t \geq 0,$$

subject to inhomogeneous boundary conditions

$$u(0,t) = -1, \quad u(2,t) = 3, \quad u(x,0) = g(x), \quad u_t(x,0) = 0$$

for the function

$$g(x) = \begin{cases} 0, & 0 \leq x < 1 \\ \sin[\pi(x-2)], & 1 \leq x < 4 \end{cases} \tag{5.23}$$

Finally, note that apart from the above trick to solve problems where the solution is constant rather than zero on the boundary, separation of variables method has been set up to solve problems that have one truly inhomogeneous boundary condition. E.g. the initial condition $u(x, 0) = h(x)$ for the heat equation, the boundary conditions $u(0, t) = f(t)$ for the wave equation, or a single inhomogeneous boundary condition $u(x, b) = f(x)$ for the Laplace equation.

What if we have two truly inhomogeneous boundary conditions, e.g. the Laplace equation, subject to

$$u(x, 0) = 0, \quad u(x, b) = f(x), \quad u(0, y) = 0, \quad u(a, y) = g(y)?$$

Well, again we can use the principle of linear superposition to solve the Laplace equation twice, for functions v and w subject to the boundary conditions:

$$v(x, 0) = 0, \quad v(x, b) = f(x), \quad v(0, y) = 0, \quad v(a, y) = 0$$

and

$$w(x, 0) = 0, \quad w(x, b) = f(x), \quad w(0, y) = 0, \quad w(a, y) = g(y).$$

Then we let $u = v + w$.

Worked example 5.9 *Solve the Laplace equation*

$$u_{xx} + u_{yy} = 0$$

subject to the inhomogeneous boundary conditions

$$u(x, 0) = 0, \quad u(x, b) = f(x), \quad u(0, y) = 0, \quad u(a, y) = g(y)$$

where the functions f and g are given by (5.21) and (5.23) respectively

5.7 Summary

- The separation of variables is a trial and error method. We TRY $u(x, t) = X(x)T(t)$ (or $u(x, y) = X(x)Y(y)$).

- The choice of the *sign* of the separation constant is crucial. The key idea is that we want to express the inhomogeneous boundary as a sum of sines or cosines, e.g:

$$\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{OR} \quad \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad .$$

- We then use half-range Fourier series to compute the coefficients a_n and b_n.
- The principle of linear superposition is our friend!

  If we cannot solve a problem we break it up into separate bits that we can solve, and then add the solutions together.