EMa2 LSPDEs 2008 **Example Sheet 1: Fourier series**

1 Sketch each of the following functions over the defined interval stated, and also their odd and even periodic extensions: **(a)** $f(t) = \cos(t), \quad 0 \le t < \pi/2$

$$\textbf{(b)} \quad f(t) = \begin{cases} t & 0 \le t < 1 \\ 2 - t & 1 \le t < 2 \end{cases} \qquad \textbf{(c)} \quad f(t) = \begin{cases} \sin(t) & 0 \le t \le \pi \\ 0 & \pi \le t < 2\pi \end{cases}$$

2 Compute the Fourier series of the following functions that have period $T = 2\pi$, and can be expressed on the fundamental interval $-\pi \le t < \pi$ as

$$\textbf{(a)} \quad f(t) = t, \qquad \textbf{(b)} \quad f(t) = |t|, \qquad \textbf{(c)} \quad f(t) = \begin{cases} 0 & -\pi \le t < 0 \\ t & 0 \le t < \pi \end{cases}$$

3 Note that the answer to 2(c) is equal to half the sum of the answers to 2(a) and 2(b). Why is that?

4 Compute the Fourier series of the following functions of have period $T = 2$ that can be expressed on the fundamental interval $-1 \le t \le 1$ as

$$\textbf{(a)} \quad f(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \end{cases} \qquad \textbf{(b)} \quad f(t) = \begin{cases} -1 & -1 \le t < 0 \\ 2 & 0 \le t < 1 \end{cases}$$

$$\textbf{(c)} \quad f(t) = \begin{cases} 0 & -1 \le t < 0 \\ t^2 & 0 \le t < 1 \end{cases}$$

5 Prove from first principles the following relations that were stated in the lecture

$$\textbf{(a)} \int_{-\pi}^{\pi} \sin(mt)\cos(nt)dt = 0 \quad \text{for all } m \text{ and } n.$$

$$\textbf{(b)} \int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \begin{cases} 0, & m \ne n \\ \pi, & m = n \end{cases}$$

6 Find the Fourier half-range cosine and sine series for the functions

$$\textbf{(a)} \quad f(t) = \sin(t), \quad 0 \le t < \pi$$

$$\textbf{(b)} \quad f(t) = \begin{cases} t & 0 \le t < a/2 \\ a - t & a/2 <\le t < a \end{cases}$$

7 Consider the following functions defined over the specified intervals. Sketch the function. Decide from the sketch whether a Fourier half-range sine or cosine series will represent the function more effectively. Find this half-range series.

$$\textbf{(a)} \quad f(t) = 2t^3 - 3t^2 + 1, \qquad 0 \le t < 1$$

$$\textbf{(b)} \quad f(t) = \begin{cases} 3t - 2t^2, & 0 \le t < 1 \\ 2 - t, & 1 <\le t < 2 \end{cases}$$

Numerical answers: **2(a)** $a_n = 0$, $b_n = 2\frac{(-1)^n}{n}$. **(b)** $a_0 = \pi$, $a_n = 2\frac{(-1)^n - 1}{\pi n^2}$, $b_n = 0$ **(c)** $a_0 = \pi/2$, $a_n = \frac{(-1)^n - 1}{\pi n^2}$, $b_n = \frac{(-1)^n}{n}$. **4(a)** $a_n = 0$, $b_n = 2\frac{1-(-1)^n}{n\pi}$. **(b)** $a_0 = 1$, $a_n = 0h$ (for $n > 1$), $b_n = 3\frac{1-(-1)^n}{n\pi}$. **(c)** $a_0 = 1/3$, $a_n = 2\frac{(-1)^n}{n^2\pi^2}$ $b_n = -\frac{-2(-1)^n + 2 + 2n^2\pi^2(-1)^n - n^2\pi^2}{n^3\pi^3}$ **6(a)** $a_0 = 4/\Pi$. $a_n = 2\frac{(-1)^n + 1}{\pi(n+1)(n-1)}$, $b_1 = 1$, $b_n = 0$ (for $n > 1$). **(b)** $a_0 = \frac{a}{2}$, $a_n = 2a\frac{(-1)^n + 1}{n^2\pi^2}$, $b_n = -4a\frac{(-1)^n}{n^2\pi^2}$. **7(a)** $a_0 = \frac{3}{2}$, $a_n = \frac{6}{n^4\pi^4}(6 + n^2\pi^2(-1)^n - 6(-1)^n)$. **(b)** $b_n = 4\frac{(-1)^n - 1}{n^3\pi^3}$