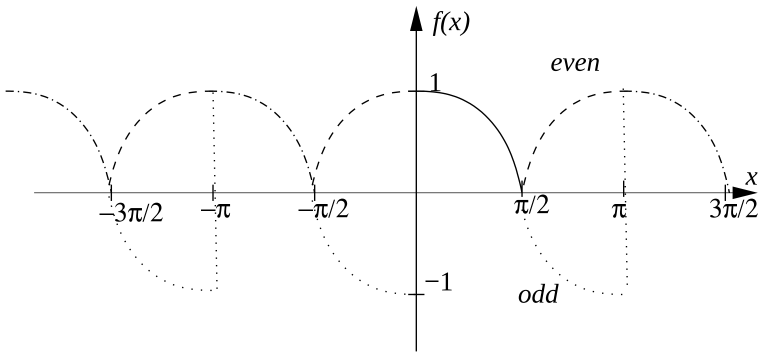
EMaII LSPDE

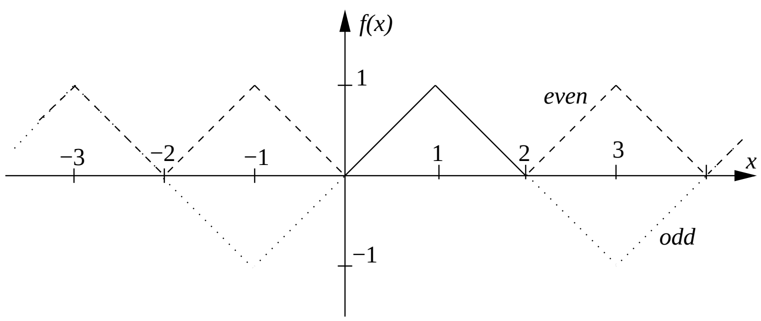
Example Sheet 1 solutions: Fourier series

2007

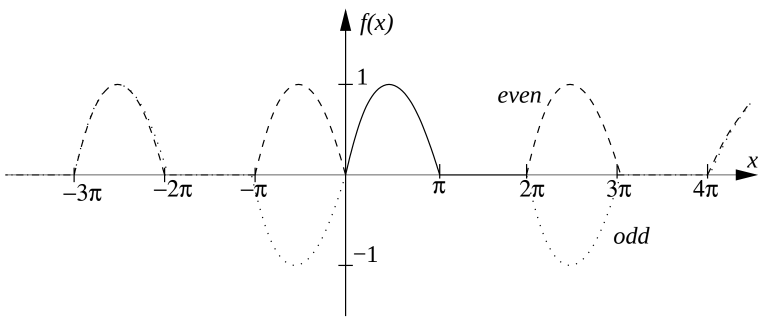
1(a)



1(b)



1(c)



2(a) We seek

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nt) + b_n \sin(nt)]$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt\,, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt\,.$$

We have $f(t) = t$ which is an odd function. Hence $a_n = 0$, including the constant term a_0. We calculate the b_n through integration by parts

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)dt \\ &= \frac{1}{\pi}\left(\left[\frac{t}{n}\cos(nt)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{n}\cos(nt)dt\right) \\ &= \frac{1}{\pi}\left[\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_{-\pi}^{\pi} \\ &= \left[\frac{1}{n}\cos(n\pi)\right] - \left[\frac{-1}{n}\cos(-n\pi)\right] \\ &= \frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^n \end{aligned}$$

2(b) This is an even function. Hence in this case we have $b_n = 0$. We get

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}|t|dt = \int_{-\pi}^{0} -\frac{t}{\pi}dt + \int_{0}^{\pi}\frac{t}{\pi}dt \\ &= \left[-\frac{t^2}{2\pi}\right]_{-\pi}^{0} + \left[\frac{t^2}{2\pi}\right]_{0}^{\pi} \\ &= +\left[\frac{\pi}{2}\right] + \left[\frac{\pi}{2}\right] \\ &= \pi\,. \end{aligned}$$

and, upon integration by parts we get

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\cos(nt)dt = \int_{-\pi}^{0} -\frac{t}{\pi}\cos(nt)dt + \int_{0}^{\pi}\frac{t}{\pi}\cos(nt)dt \\ &= \left[-\frac{t}{n\pi}\sin(nt)\right]_{-\pi}^{0} + \int_{-\pi} 0\frac{1}{n\pi}\sin(nt)dt + \left[\frac{t}{n\pi}\sin(nt)\right]_{0}^{\pi} - \int_{0}\pi\frac{1}{n\pi}\sin(nt)dt \\ &= -\left[\frac{t}{n\pi}\sin(nt) + \frac{1}{n^2\pi}\cos(nt)\right]_{-\pi}^{0} + \left[\frac{t}{n\pi}\sin(nt) + \frac{1}{n^2\pi}\cos(nt)\right]_{0}^{\pi} \\ &= \frac{1}{n^2\pi^2}(-[1] + [\cos(n\pi)] + [\cos(n\pi)] - [1]) \\ &= 2\frac{(-1)^n - 1}{\pi n^2} \end{aligned}$$

2(c)

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \\ &= \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt \\ &= \frac{1}{\pi}\left(\left[\frac{t}{n}\cos(nt)\right]_{0}^{\pi} - \int_{0}^{\pi}\frac{1}{n}\cos(nt)dt\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\left[\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_0^\pi \\
&= \frac{1}{n}\cos(n\pi) = \frac{(-1)^n}{n}
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \\
&= \frac{1}{\pi}\int_0^{\pi} tdt \\
&= \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^\pi \\
&= \frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \\
&= \frac{1}{\pi}\int_0^{\pi} t\sin(nt)dt \\
&= +\left[\frac{t}{n\pi}\sin(nt)\right]_0^\pi - \int_0 \pi\frac{1}{n\pi}\sin(nt)dt \\
&= \left[\frac{t}{n\pi}\sin(nt) + \frac{1}{n^2\pi}\cos(nt)\right]_0^\pi \\
&= \frac{1}{n^2\pi^2}(-1+\cos(nt)) = \frac{1}{n^2\pi^2}((-1)^n - 1)
\end{aligned}
$$

3 Note that the function in part (c) is $(1/2)(|t| + t)$. Therefore its Fourier series should be the sum of half the Fourier series of $|t|$ and t.

4(a) This is an odd function, so $a_n = 0$. Hence we only need to compte the b_n's.

$$
b_n = \frac{2}{T}\int_{-1}^{1} f(t)\sin(n\omega t)
$$

where $T = 2$ and $\omega = \frac{2\pi}{2} = \pi$. Hence

$$
\begin{aligned}
b_n &= \int_{-1}^{1} f(t)\sin(n\pi t)dt \\
&= \int_{-1}^{0} -\sin(n\pi t)dt + \int_0^1 \sin(n\pi t)dt \\
&= \left[\frac{1}{n\pi}\cos(n\pi t)\right]_{-1}^0 - \left[\frac{1}{n\pi}\cos(n\pi t)\right]_0^1 \\
&= \frac{1}{n\pi}([1] - [\cos(-n\pi)] - [\cos(n\pi)] + [1]) \\
&= \frac{1}{n\pi}(1 - (-1)^n)
\end{aligned}
$$

4(b)

$$
\begin{aligned}
a_0 &= \int_{-1}^{1} f(t)dt \\
&= \int_{-1}^{0} -1dt + \int_0^1 2dt \\
&= [-t]_{-1}^0 + [2t]_0^1 \\
&= -1 + 2 = 1
\end{aligned}
$$

$$\begin{aligned}
a_n &= \int_{-1}^{0} -\cos(n\pi t)dt + \int_{0}^{1} 2\cos(n\pi t)dt \\
&= \frac{1}{n\pi}\left([-\sin(n\pi t)]_{-1}^{0} + [2\sin(n\pi t)]_{0}^{2}\right) \\
&= 0 \text{ because } \sin(m\pi) = 0 \text{ for all } m
\end{aligned}$$

$$\begin{aligned}
b_n &= \int_{-1}^{0} -\sin(n\pi t)dt + \int_{0}^{1} 2\sin(n\pi t)dt \\
&= \frac{1}{n\pi}\left([\cos(n\pi t)]_{-1}^{0} + [-2\cos(n\pi t)]_{0}^{2}\right) \\
&= \frac{1}{n\pi}([1] - [\cos(-n\pi)] - [2\cos(-n\pi)] + [2]) \\
&= \frac{3}{n\pi}(1 - (-1)^n)
\end{aligned}$$

4(c)

$$\begin{aligned}
a_0 &= \int_{-1}^{1} f(t)dt \\
&= \int_{0}^{1} t^2 dt \\
&= \left[\frac{t^3}{3}\right]_{0}^{1} \\
&= 1/3
\end{aligned}$$

$$\begin{aligned}
a_n &= \int_{0}^{1} t^2\cos(n\pi t)dt \\
&= [(t^2/n\pi)\sin(n\pi t)]_0^1 - \int_0^1 (2t/n\pi)\sin(n\pi t)dt \\
&= [(t^2/n\pi)\sin(n\pi t)]_0^1 + [(2t/n^2\pi^2)\cos(n\pi t)]_0^1 - \int_0^1 (2/n^2\pi^2)\cos(n\pi t)dt \\
&= [(t^2/n\pi)\sin(n\pi t) + (2t/n^2\pi^2)\cos(n\pi t) - (2/n^3\pi^3)\sin(n\pi t)]_0^1 \\
&= [(2/n^2\pi^2)\cos(n\pi)] - [0] = 2\frac{(-1)^n}{n^2\pi^2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \int_{0}^{1} t^2\sin(n\pi t)dt \\
&= [-(t^2/n\pi)\cos(n\pi t)]_0^1 + \int_0^1 (2t/n\pi)\cos(n\pi t)dt \\
&= [-(t^2/n\pi)\sin(n\pi t)]_0^1 + [(2t/n^2\pi^2)\sin(n\pi t)]_0^1 - \int_0^1 (2/n^2\pi^2)\sin(n\pi t)dt \\
&= [-(t^2/n\pi)\cos(n\pi t) + (2t/n^2\pi^2)\sin(n\pi t) + (2/n^3\pi^3)\cos(n\pi t)]_0^1 \\
&= [-(1/n^2\pi)\cos(n\pi) + (2/n^3\pi^3)\cos(n\pi)] - [2/n^3\pi^3] \\
&= -\frac{-2(-1)^n + 2 + 2n^2\pi^2(-1)^n - n^2\pi^2}{n^3\pi^3}
\end{aligned}$$

5(a)

$$\begin{aligned}
\int_{-\pi}^{\pi} \sin(mt)\cos(nt)dt &= \int_{-\pi}^{\pi} \frac{1}{2}(\sin[(m+n)t] + \sin[(m-n)t])dt \\
&= \left[-\frac{1}{2}\left(\frac{1}{m+n}\cos(m+n)t + \frac{1}{m-n}\cos(m-n)t\right)\right]_{-\pi\pi} \\
&= 0 \quad \text{provided } m \neq n.
\end{aligned}$$

If $m = n$ then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mt)\cos(mt)dt &= \int_{-\pi}^{\pi} \frac{-1}{2}\sin(2mt)dt \\ &= \left[\frac{1}{2}\frac{1}{2m}\cos(2mt)\right]_{-\pi^{\pi}} \\ &= 0\end{aligned}$$

□

5(b)

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt &= \int_{-\pi}^{\pi} \frac{1}{2}(\cos[(m+n)t] + \cos[(m-n)t])dt \\ &= \left[\frac{1}{2}\left(\frac{1}{m+n}\sin(m+n)t + \frac{1}{m-n}\sin(m-n)t)\right)\right]_{-\pi^{\pi}} \\ &= 0 \quad \text{provided } m \neq n.\end{aligned}$$

If $m = n$ then

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2(mt)dt &= \int_{-\pi}^{\pi} \frac{1}{2}(1+\cos(2mt))dt \\ &= \left[\frac{1}{2}\left(t + \frac{1}{2m}\sin(2m)t\right)\right]_{-\pi^{\pi}} \\ &= \pi\end{aligned}$$

□

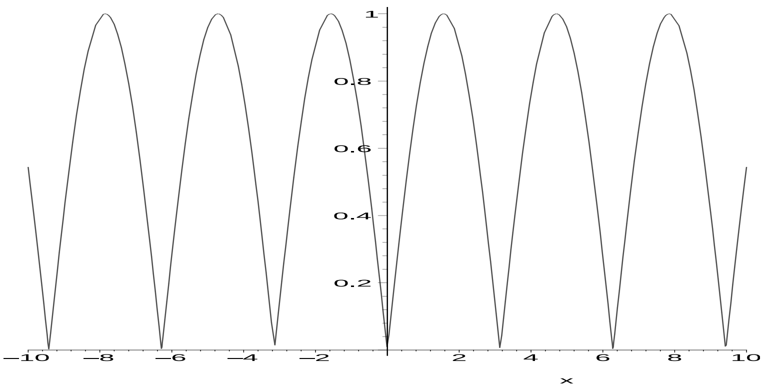
6(a) To find the Fourier $\sin(t)$ series we set use $L = \pi$ and set

$$\begin{aligned}b_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nt)\sin(t)dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(\cos[(n-1)t] + \cos[(n+1)t]\right)dt \\ &= \frac{1}{\pi}\left[\frac{1}{n-1}\sin[(n-1)t] + \frac{1}{n+1}\sin[(n+1)t]\right]_0^{\pi} \\ &= 0 \quad \text{provided } n \neq 1.\end{aligned}$$

If $n = 1$ we have

$$\begin{aligned}b_1 &= \frac{2}{\pi}\int_0^{\pi} \sin^2(t)dt \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\left(1+\cos(2t)\right)dt \\ &= \frac{1}{\pi}[t + (1/2)\cos(2t)]_0^{\pi} \\ &= 1\end{aligned}$$

Now one might imagine that the cosine seris is zero. But this is not true. This is because we are taking a half-range cosine series which is really the Fourier approtimation to the *even* function $|\sin(t)|$



So we get

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi \sin(t)dt \\ &= \frac{2}{\pi}[-\cos(t)]_0^\pi \\ &= \frac{4}{\pi}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \sin(t)\cos(nt)dt \\ &= \frac{1}{\pi}\int_0^\pi \sin[(n+1)t] - \sin[(n-1)t]dt \\ &= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos[(n+1)t] + \frac{1}{n-1}\cos[(n-1)t]\right]_0^\pi dt \\ &= \frac{1}{\pi}\left(\left[\frac{1}{n-1} - \frac{1}{n+1}\right] - \left[\left(\frac{1}{n-1} - \frac{1}{n+1}\right)(-1)^{n+1}\right]\right) \\ &= \frac{2}{\pi(n-1)(n+1)}[1+(-1)^n] \end{aligned}$$

6(b) Here the function is specified on the interval $L = a$ and so $\omega = \frac{\pi}{L} = \frac{\pi}{a}$. So for the cosine half-range etpansion we get

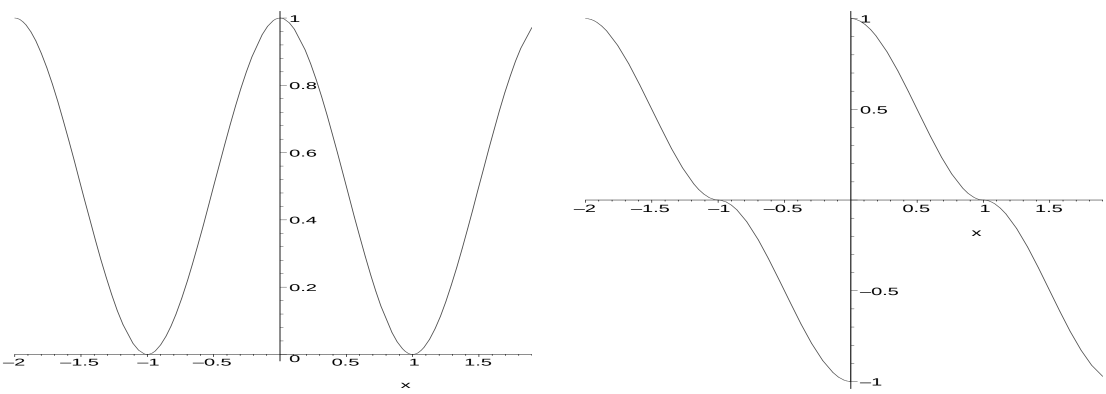
$$\begin{aligned} a_0 &= \frac{2}{a}\int_0^a f(t)dt \\ &= \frac{2}{a}\int_0^{a/2} tdt + \frac{2}{a}\int_{a/2}^a a - tdt \\ &= \frac{2}{a}[t^2/2]_0^{a/2} + \frac{2}{a}[at - t^2/2]_{a/2}^a \\ &= \frac{2}{a}\left(a^2 8 + \frac{a^2}{4} - \frac{a^2}{8}\right) \\ &= \frac{a}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{a}\int_0^a f(t)\cos\left(\frac{n\pi t}{a}\right)dt \\ &= \frac{2}{a}\int_0^{a/2} t\cos\left(\frac{n\pi t}{a}\right)dt\frac{2}{a}\int_{a/2}^a (a-t)\cos\left(\frac{n\pi t}{a}\right)dt \\ &= \frac{2}{a}\left(\left[\frac{at}{n\pi}\sin\left(\frac{n\pi t}{a}\right)\right]_0^{a/2} - \int_0^{a/2}\frac{a}{n\pi}\sin\left(\frac{n\pi t}{a}\right)dt\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{a}\left(\left[\frac{a(a-t)}{n\pi}\sin\left(\frac{n\pi t}{a}\right)\right]_{a/2}^{a}+\int_{a/2}^{a}\frac{a}{n\pi}\sin\left(\frac{n\pi t}{a}\right)dt\right)\\
=\;&\frac{2}{a}\left[\frac{at}{n\pi}\sin\left(\frac{n\pi t}{a}\right)+\frac{a^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{a}\right)\right]_0^{a/2}\\
&++\frac{2}{a}\left[\frac{a(a-t)}{n\pi}\sin\left(\frac{n\pi t}{a}\right)-\frac{a^2}{n^2\pi^2}\cos\left(\frac{n\pi t}{a}\right)\right]_{a/2}^{a}\\
=\;&\left[\frac{a}{n\pi}\sin(n\pi/2)\right]-\left[\frac{2a}{n^2\pi^2}\right]-\left[\frac{2a}{n^2\pi^2}\cos(n\pi)\right]-\left[\frac{a}{n\pi}\sin(n\pi/2)\right]\\
=\;&\frac{-2a}{n^2\pi^2}(1+(-1)^n)
\end{aligned}
$$

$$
\begin{aligned}
b_n =\;&\frac{2}{a}\int_0^a f(t)\sin\left(\frac{n\pi t}{a}\right)dt\\
=\;&\frac{2}{a}\int_0^{a/2} t\sin\left(\frac{n\pi t}{a}\right)dt\frac{2}{a}\int_{a/2}^{a}(a-t)\sin\left(\frac{n\pi t}{a}\right)dt\\
=\;&\frac{2}{a}\left(\left[-\frac{at}{n\pi}\cos\left(\frac{n\pi t}{a}\right)\right]_0^{a/2}+\int_0^{a/2}\frac{a}{n\pi}\cos\left(\frac{n\pi t}{a}\right)dt\right)\\
&-\frac{2}{a}\left(\left[\frac{a(a-t)}{n\pi}\cos\left(\frac{n\pi t}{a}\right)\right]_{a/2}^{a}+\int_{a/2}^{a}\frac{a}{n\pi}\cos\left(\frac{n\pi t}{a}\right)dt\right)\\
=\;&\frac{2}{a}\left[-\frac{at}{n\pi}\cos\left(\frac{n\pi t}{a}\right)+\frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi t}{a}\right)\right]_0^{a/2}\\
&-\frac{2}{a}\left[\frac{a(a-t)}{n\pi}\cos\left(\frac{n\pi t}{a}\right)+\frac{a^2}{n^2\pi^2}\sin\left(\frac{n\pi t}{a}\right)\right]_{a/2}^{a}\\
=\;&\left[-\frac{a}{n\pi}\cos(n\pi/2)\right]+\left[\frac{2a}{n^2\pi^2}\right]-\left[\frac{2a}{n^2\pi^2}\cos(n\pi)\right]-\left[\frac{a}{n\pi}\cos(n\pi/2)\right]\\
=\;&\frac{-4a}{n^2\pi^2}((-1)^n)
\end{aligned}
$$

7(a) Plotting the even (left-hand plot) and odd (right-hand) extensions of f



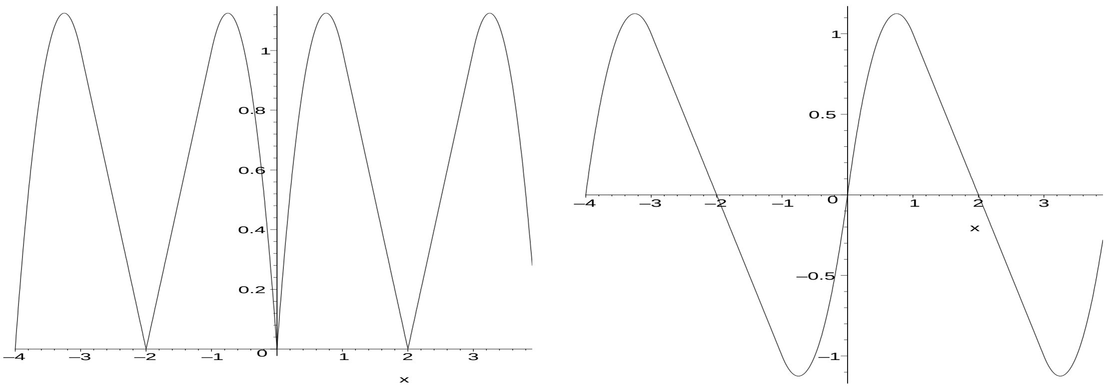
we see immediately that the even extension leads to a smooth function, whereas the odd extension leads to a function with jumps (discontinuities). Recalling that the convergence of Fourier series is slower at discontinuitites and leads to the Gibb's phenomenon (overshoot), we would naturally choose to represent the odd extension of the function. Hence we opt for a Fourier cosine series.

Here we have $L = 1$ and $\omega = \pi/L = \pi$. Hence we have

$$a_0 = 2\int_0^1 (2t^3 - 3t^2 + 1)dt = [(t^4/2) - t^3 + t]_0^1 = 1$$

$$\begin{aligned}
a_n &= \int_0^1 (2t^3 - 3t^2 + 1)\cos(n\pi t)dt \\
&= \text{after integration by parts 3 times} \\
&= \Bigg[2\,\frac{n^3\pi^3 t^3 \sin(n\,\pi\,t) + 3\,n^2\pi^2 t^2 \cos(n\,\pi\,t) - 6\,\cos(n\,\pi\,t) - 6\,n\,\pi\,t\sin(n\,\pi\,t)}{n^3\pi^3} - 3\,\frac{n^2\pi^2 t^2 \sin(n\,\pi \\
&= \text{after realising that most of these terms evaluate to 0} \\
&= a_n = 24\frac{1-(-1)^n}{n^4\pi^4}
\end{aligned}$$

7(b) Plotting the even (left-hand plot) and odd (right-hand) extensions of f



we see immediately that the even extension leads to a function with corners (jumps in the slope of the graph) whereas the odd extension leads to a smooth function. Recalling that the convergence of Fourier series is slower at sharp corners, we would naturally choose to represent the odd extension of the function. Hence we opt for a Fourier sine series.

Here $L = 2$ so $\omega = \pi/L = \pi/2$

Hence

$$\begin{aligned}
b_n &= \int_0^2 f(t)\sin(n\pi t/2)dt \\
&= \int_0^1 (3t - 2t^2)\sin(n\pi t/2)dt + \int_1^2 (2-t)\sin(n\pi t/2)dt \\
&= +2\left[\frac{-2\,n\,\pi\,\cos(1/2\,n\,t\pi) - 2\,\sin(1/2\,n\,t\pi) + n\,t\pi\,\cos(1/2\,n\,t\pi)}{n^2\pi^2}\right]_0^1 \\
&\quad \left[2\,\frac{-2\,n\,\pi\,\cos(1/2\,n\,\pi\,t) - 2\,\sin(1/2\,n\,\pi\,t) + n\,\pi\,t\cos(1/2\,n\,\pi\,t)}{n^2\pi^2}\right]_1^2 \\
&= 32\frac{(1-\cos(n\pi/2))}{n^3\pi^3}
\end{aligned}$$