

Worked example 1.3

We have the function $f(t) = 1 - |t|$ on the interval $-1 \leq t \leq 1$ and we are asked to find the fourier series expansion. Note that this function is even, so if we think carefully we can realize that there should be no b_n terms, because the sin function is odd.

We have

$$\begin{aligned} a_0 &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt \\ &= 1 \int_{-1}^1 (1 - |t|) dt = \int_{-1}^0 (1 + t) dt + \int_0^1 (1 - t) dt = 1 \end{aligned}$$

(you can do this using the graph too; it's just the area of two triangles, which we drew in class). So

$$a_0 = 1.$$

Now we need a_n , which are given by

$$\begin{aligned} a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t) dt \\ &= \int_{-1}^1 (1 - |t|) \cos(n\pi t) dt \\ &= \int_{-1}^0 (1 + t) \cos(n\pi t) dt + \int_0^1 (1 - t) \cos(n\pi t) dt \\ &= \int_{-1}^1 \cos(n\pi t) dt + \int_{-1}^0 t \cos(n\pi t) dt + \int_0^1 -t \cos(n\pi t) dt \end{aligned}$$

The first of these integrals is 0. The latter two have to be done by parts but we can save a little time by realizing that they are very similar to each other: let $s = -t$ and $ds = -dt$ in the middle integral. Then it becomes

$$\int_{-1}^0 t \cos(n\pi t) dt \rightarrow \int_{s=1}^{s=0} (-s) \cos(n\pi(-s))(-ds) = - \int_0^1 s \cos(n\pi s) ds$$

using the fact that \cos is even, and swapping the sign when we swap the lower and upper limits (to integrate from 0 to 1 rather than 1 to 0).

We put this into the expression for a_n to get

$$\begin{aligned} a_n &= -2 \int_0^1 t \cos(n\pi t) dt \\ &= -2 \left(\left[\frac{t}{n\pi} \sin(n\pi s) \right]_0^1 - \int_0^1 \frac{1}{n\pi} \sin(n\pi s) ds \right) \\ &= 2 \int_0^1 \frac{1}{n\pi} \sin(n\pi s) ds = \frac{2}{n\pi} \frac{1}{n\pi} [-\cos(n\pi s)]_0^1 \\ &= \frac{2}{(n\pi)^2} [-\cos(n\pi) + \cos(0)] \\ &= \frac{2}{(n\pi)^2} [1 - \cos(n\pi)]. \end{aligned}$$

This expression is only nonzero when n is odd because $\cos(n\pi) = (-1)^n$ (which you should check for yourself!). So the bit in the brackets is either 0 or 2. Furthermore, any odd number can be written as $2m - 1$ for some integer m . This gives the expression in the main lecture notes, ie

$$\begin{aligned} f(t) &\sim \frac{1}{2} + \frac{4}{\pi^2} \cos(\pi t) + \frac{4}{9\pi^2} \cos(3\pi t) + \frac{4}{25\pi^2} \cos(5\pi t) + \dots \\ &= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2\pi^2} \cos[(2m-1)\pi t] \end{aligned}$$

You should do the same procedure for the b_n terms and get 0 as an answer, because the function is even.

Worked example 1.4

We have

$$f(t) = \begin{cases} 1 & -L < t < 0 \\ (1 - t/L), & 0 < t < L \end{cases}$$

on its fundamental domain, and period $T = 2L$. This gives

$$\begin{aligned} a_0 &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt \\ &= \frac{1}{L} \int_{-L}^L f(t) dt \\ &= \frac{1}{L} \left[\int_{-L}^0 dt + \int_0^L (1 - t/L) dt \right] \\ &= \frac{3}{2}. \end{aligned}$$

We have

$$\begin{aligned} a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n\omega t) dt \\ &= \frac{1}{L} \left[\int_{-L}^0 \cos\left(\frac{n\pi t}{L}\right) dt + \int_0^L (1 - t/L) \cos\left(\frac{n\pi t}{L}\right) dt \right] \\ &= \frac{1}{L} \int_{-L}^L \cos\left(\frac{n\pi t}{L}\right) dt - \frac{1}{L} \int_0^L \frac{t}{L} \cos\left(\frac{n\pi t}{L}\right) dt \\ &= 0 - \frac{1}{L^2} \int_0^L t \cos\left(\frac{n\pi t}{L}\right) dt \end{aligned}$$

We do this integral by parts. It's very similar to the one in worked example 1.3, above. We get

$$\begin{aligned}
a_n &= -\frac{1}{L^2} \left(\left[t \frac{L}{n\pi} \sin\left(\frac{n\pi t}{L}\right) \right]_0^L - \int_0^L \frac{L}{n\pi} \sin\left(\frac{n\pi t}{L}\right) dt \right) \\
&= \text{do these details yourself...} \\
&= \frac{-1}{(\pi n)^2} (\cos(\pi n) - \cos(0)) \\
&= \frac{1}{(\pi n)^2} [1 - (-1)^n]
\end{aligned}$$

Now we need the b_n . The function f in this example is neither even nor odd, so neither the a_n 's or the b_n 's will all be 0. We have $T = 2L$, $\omega = 2\pi/T = \pi/L$ so we have

$$\begin{aligned}
b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n\omega t) dt \\
b_n &= \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt \\
&= \frac{1}{L} \int_{-L}^0 \sin\left(\frac{n\pi t}{L}\right) dt + \frac{1}{L} \int_0^L (1 - t/L) \sin\left(\frac{n\pi t}{L}\right) dt \\
&= \frac{1}{L} \int_{-L}^L \sin\left(\frac{n\pi t}{L}\right) dt - \frac{1}{L^2} \int_0^L t \sin\left(\frac{n\pi t}{L}\right) dt
\end{aligned}$$

The first of these integrals is 0 and the second one we again do by parts. We get

$$\begin{aligned}
b_n &= -\frac{1}{L^2} \left(\left[t \frac{-L}{n\pi} \cos\left(\frac{n\pi t}{L}\right) \right]_0^L - \int_0^L \frac{-L}{n\pi} \cos\left(\frac{n\pi t}{L}\right) dt \right) \\
&= -\frac{1}{L^2} \left(\frac{-L^2}{n\pi} \cos(n\pi) + \frac{-L^2}{(\pi n)^2} (0) \right) \\
&= \frac{1}{n\pi} \cos(n\pi) = \frac{(-1)^n}{n\pi}.
\end{aligned}$$

To find the fourier series that the question asked for, we pull this all together into the definition of fourier series, i.e.

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

so that we have

$$f(t) \sim \frac{3}{4} + \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2} [1 - (-1)^n] \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi t}{L}\right).$$

Sketch for worked example 1.6 which uses the *half range* series, as defined in the notes.

We have the function

$$f(t) = \begin{cases} t & 0 \leq t \leq 2 \\ 4 - t & 2 \leq t \leq 4 \end{cases}.$$

The constant term a_0 is the area under $f(t)$ between 0 and L , which is 4.

We now have, from the definition of the half range cosine series,

$$a_n = \frac{2}{L} \int_0^L f_e(t) \cos\left(\frac{n\pi t}{L}\right) dt$$

which was derived in the notes. $f_e(t)$ is the even periodic extension of the function f . Here $L = 4$. Putting in f and splitting the integral over the 2 ranges given in f 's definition we have

$$a_n = \frac{1}{2} \int_0^2 t \cos\left(\frac{n\pi t}{4}\right) + \frac{1}{2} \int_2^4 (4 - t) \cos\left(\frac{n\pi t}{4}\right)$$

Again we have these integrals with $t \cos(\text{stuff})$ which we will do by parts. This should be looking familiar by now!

You can do this however you like. One option would be to write the above as 3 integrals, two of which have to be done by parts. Or you can use this clever little transformation: let $s = 4 - t$, and then $ds = -dt$, in the second integral. You get

$$\frac{1}{2} \int_2^4 (4 - t) \cos\left(\frac{n\pi t}{4}\right) = \frac{1}{2} \int_{s=2}^{s=0} s \cos\left(\frac{n\pi(4 - s)}{4}\right) (-ds).$$

Now you can use the trigonometric angle addition formulas, which give us

$$\cos\left(n\pi - \frac{n\pi s}{4}\right) = \cos(n\pi) \cos\left(\frac{n\pi s}{4}\right)$$

because the sin terms are 0

$$= \begin{cases} \cos\left(\frac{n\pi s}{4}\right) & n \text{ even} \\ -\cos\left(\frac{n\pi s}{4}\right) & n \text{ odd} \end{cases}$$

Either way, we end up with a $s \cos\left(\frac{n\pi s}{4}\right)$ being integrated between 0 and 2, which is the same as the first integral in the expression for a_n . Ok, so maybe that wasn't easier than integrating by parts twice. Anyway, we have

$$a_n = \left(\frac{1}{2} + \frac{(-1)^n}{2}\right) \int_0^2 s \cos\left(\frac{n\pi s}{4}\right) ds$$

which we still have to integrate by parts. The answer at the end is

$$a_n = \left(\frac{1}{2} + \frac{(-1)^n}{2}\right) \left(\frac{8}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \left(\frac{4}{n\pi}\right)^2\right)$$