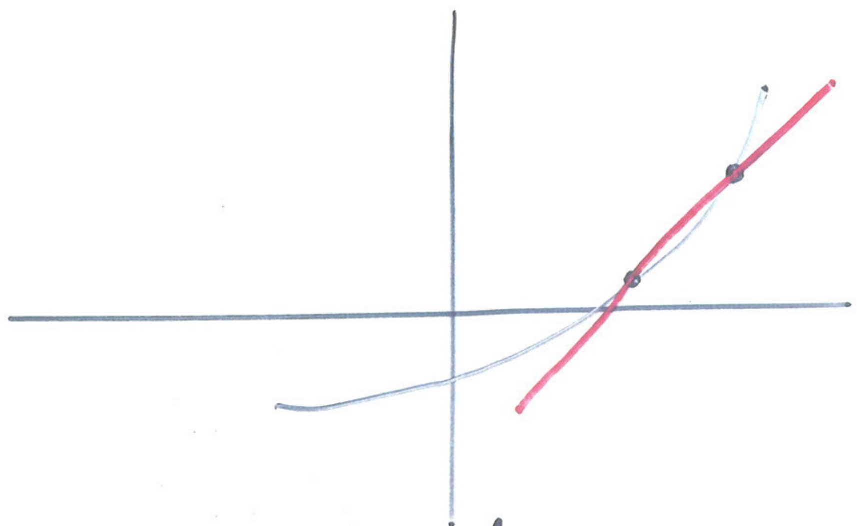
Polynomial Interpolation – KEY RESULTS

Interpolation is the art of

Reading between the Lines



A unique straight line passes through two points on a curve. We use this property to 'interpolate' a table.

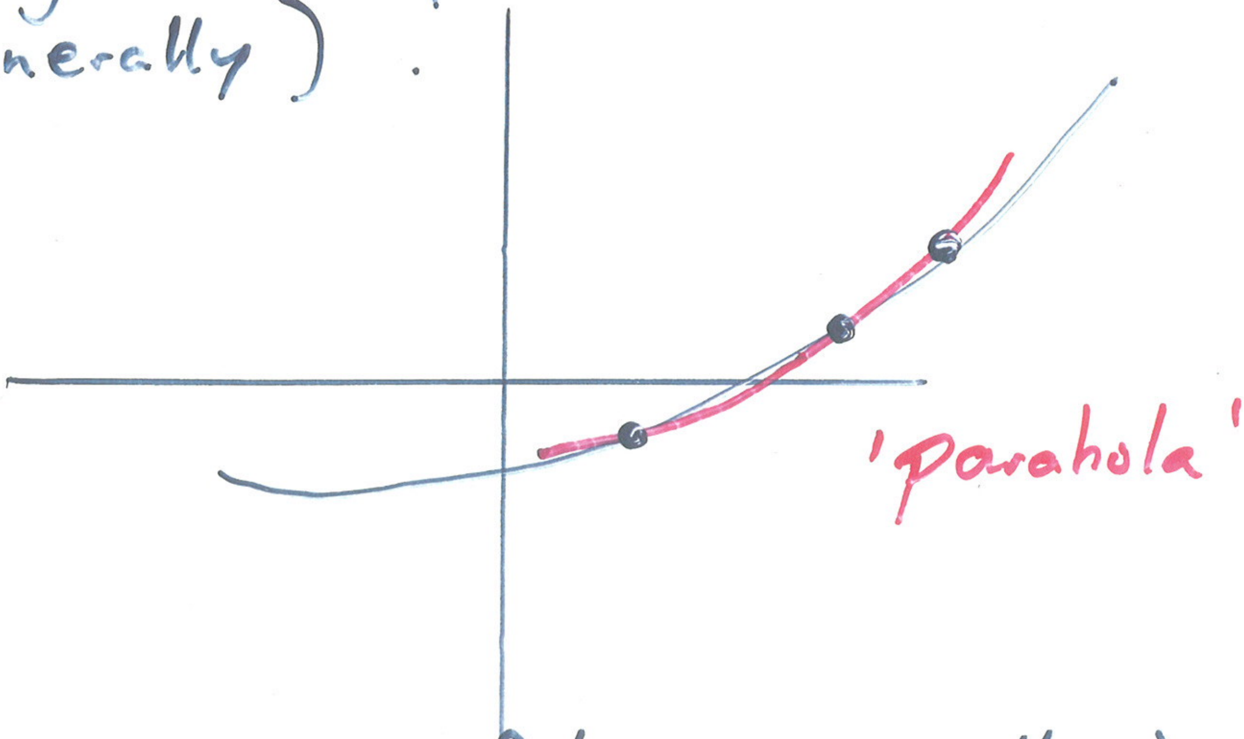
viz JAMES – MEM, p1019 – Standard Normal Curve

z	$\Phi(z)$
1.00	0.8413
1.01	0.8438
1.02	0.8461
1.03	0.8485

$\Phi(1.023)$?

≈ 0.8469.

A unique quadratic passes through three points on a curve (generally).



giving a closer fit, especially in the middle of the range.

In general a unique polynomial of degree n:

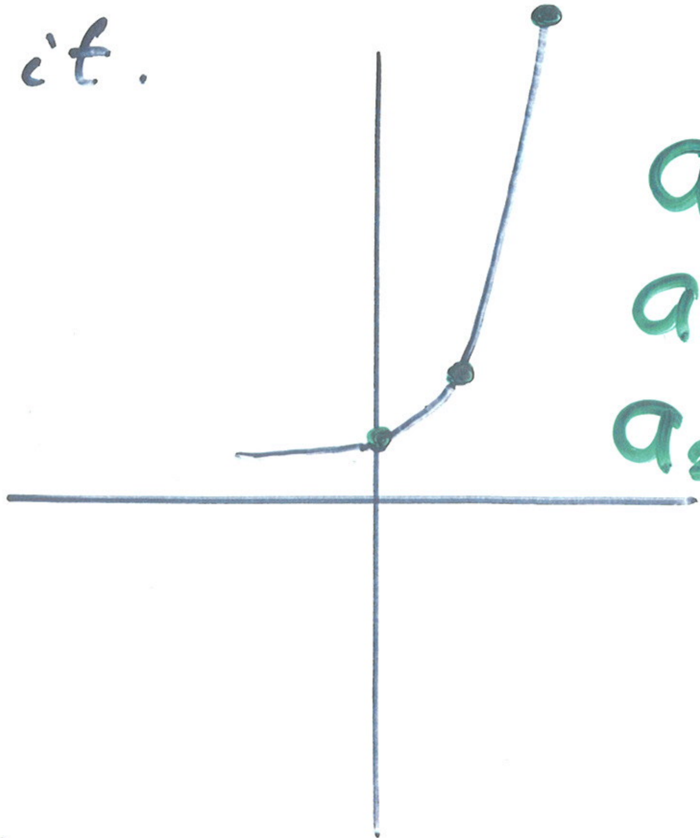
$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

passes through $n+1$ points on a curve

Polynomial Interpolation

A curve passes through (0, 1), (1, 2) and (2, 7).

Determine the equation of a quadratic to fit it.



$a_0 = 1$

$a_1 = -1$

$a_2 = 2$

We seek

$$y(x) = a_2 x^2 + a_1 x + a_0$$

$$y(0) = 0 + 0 + a_0 = 1$$

$$y(1) = a_2 + a_1 + a_0 = 2$$

$$y(2) = 4a_2 + 2a_1 + a_0 = 7$$

What are a_0, a_1, a_2?

Simple elimination delivers a_0, a_1, a_2 in this case but generally, a system of equations needs to be solved.

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

In the above case:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

Use Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 6 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right) \quad R_3'' = R_3' - 2R_2'$$

Now in upper triangular form, so back-substitute to give

$$a_2 = 2, \; a_1 = -1, \; a_0 = 1$$

So $p_2(x) = 2x^2 - x + 1$ VERIFY

Use of this polynomial will interpolate for 'x' within the range $[0, 2]$,

e.g. $x = 1.2$

$$P_2(1.2)$$

$$= \big(2(1.2) - 1\big)(1.2) + 1\big)$$

$$= (2.4 - 1)(1.2) + 1 = 2.68\,?$$

using Nested multiplication.

Lagrange Polynomial

The quadratic $L_2(x)$ passing through $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is:

$$L_2(x) = \frac{y_1 (x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_3)(x - x_1)}{(x_2 - x_3)(x_2 - x_1)}$$

$$+ y_3 \frac{(x - x_1)(x - x_2)}{(x_2 - x_1)(x_3 - x_2)}$$

So if we have $(1, 0), (1.5, 0.4), (2, 0.69)$

$$L_2(x) = 0 + 0.4 \frac{(x-2)(x-1)}{(1.5-2)(1.5-1)} + 0.69 \frac{(x-1)(x-1.5)}{(2-1)(2-1.5)}$$

Note $(x-1)$ is a factor! WHY?

What is $L_2(x)$?