Solutions to Example Sheet 3, 4, 9-12 (Extras)

Outline only (Numerical Methods)

Note these are intended as challenging problems.

(1) (a) The trapezoidal method reads

$$\frac{x_{n+1} - x_n}{h} = \frac{1}{2}f(x_n) + \frac{1}{2}f(x_{n+1})$$

For $f(x) = x^2$ we have

$$\frac{x_{n+1} - x_n}{h} = \frac{1}{2}x_n^2 + \frac{1}{2}x_{n+1}^2$$

or, rewriting,

$$x_{n+1} = \frac{h}{2}(x_n^2) + \frac{h}{2}x_{n+1}^2 + x_n$$

This is a quadratic equation for x_{n+1} in terms of x_n.

(b) We rewrite the equation for x_{n+1}:

$$\frac{h}{2}x_{n+1}^2 - x_{n+1} + \frac{h}{2}x_n^2 + x_n = 0.$$

Soln:
$$x_{n+1} = \left(\frac{2}{h}\right)\left[-1 \pm \sqrt{1 - \frac{4h}{2}\left(\frac{h}{2}x_n^2 + x_n\right)}\right]$$

The "natural" choice is the positive root, because we know that $\frac{dx}{dt} > 0$ so x is increasing, and $x(0) = 1$ which is positive.

(c) Here the question asks way to use Newton's method to solve

$$\frac{h}{2} x_{n+1}^2 - x_{n+1} + \frac{h}{2} x_n^2 + x_n = 0$$

rather than the quadratic formula.

It's not hard but we must choose notation to make it clear how to do this.

Let y^i denote the Newton's method iterates so that

$$y^{i+1} = y^i - \frac{g(y^i)}{g'(y^i)}$$

Here $g(y^i) = \frac{h}{2}(y^i)^2 - y^i + \frac{h x_n^2}{2} + x_n$

(which is the thing we need to be $= 0$)

So we start with $x_0 = x(0) = 1$

and y^0 = initial guess for Newton's method $= x_0$

So $y^1 = y^0 - \frac{g(y^0)}{g'(y^0)}$: set $h = 0.2$ in $g(y)$

$$y^2 = y^1 - \frac{g(y^1)}{g'(y^1)}, \quad y^3 = y^2 - \frac{g(y^2)}{g'(y^2)}$$

y^3 is an approximation of $x(0.2)$.

(d) [Sketch] Instead of starting with the trapezoidal formula start with $\frac{x_{n+1} - x_n}{h} = x_{n+1}^2$, again quadratic for x_{n+1}.

9. (a) Midpoint rule

$$\frac{x_{n+1} - x_{n-1}}{2h} = f(x_n)$$

(*) $x_{n+1} = 2h\, f(x_n) + x_{n-1}$

We'll need x_0 and x_1 to find x_2.

Use Euler's method to get a guess for x_1:

$x_1 = x_0 + h\, f(x_0, t_0)$.

$x_1 = 0 + 0.1\,(-1) = -0.1$.

Then use (*)

$$x_2 = 2(0.1)\left[-12(-0.1) + \frac{3}{2}t_1^2 - 12t_1 + 6t_1^3 - 1\right] + x_1$$

where $t_1 = 0.1$

$$x_3 = 2(0.1)\left[-12(x_2) + \frac{3}{2}t_2^2 - 12t_2 + 6t_2^3 - 1\right] + x_2$$

$t_2 = 0.2$

⋮

and so on, to $x_{10} \approx x(1)$.

(b) Predictor-corrector

$$x_{n+1} = x_n + \frac{h}{2} f(x_n, t_n) + \frac{h}{2} f(\underbrace{x_n + h\, f(x_n, t_n)}_{\text{call this "corrector" } y_n}, t_{n+1})$$

So at each stage we can compute

$$y_n = x_n + hf(x_n, t_n)$$

and then

$$x_{n+1} = x_n + \frac{h}{2} f(x_n, t_n) + \frac{h}{2} f(y_n, t_{n+1})$$

Example

$$y_0 = x_0 + hf(x_0, t_0) = 0 + 0.1(-1) = -0.1.$$

$$x_1 = x_0 + \frac{h}{2} f(x_0, t_0) + \frac{h}{2} f(y_0, t_1)$$

$$= 0 + (0.05)(-1) + (0.05)\left[-12(-0.1) + \frac{3}{2}t_1^2 - 12t_1 + 6t_1^3 - 1\right]$$

with $t_1 = 0.1$.

Then

$$y_1 = x_1 + hf(x_1, t_1)$$

$$= x_1 + (0.1)\left[-12x_1 + \frac{3}{2}t_1^2 - 12t_1 + 6t_1^3 - 1\right]$$

and

$$x_2 = x_1 + \frac{h}{2} f(x_1, t_1) + \frac{h}{2} f(y_1, t_2)$$

and so on.

The predictor-corrector method will be better than the midpoint rule because it doesn't implicitly assume the linearity in averaging $\frac{1}{h}(x_{n+1} - x_{n-1}) = f(x_n)$

10. This question requires the non-autonomous Runge-Kutta method, but the lecture notes only give the autonomous version. For reference see James v2 pp 651–654.

We have $X_{n+1} = X_n + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4)$

where
$$c_1 = h f(t_n, X_n)$$
$$c_2 = h f(t_n + \tfrac{1}{2}h, X_n + \tfrac{1}{2}c_1)$$
$$c_3 = h f(t_n + \tfrac{1}{2}h, X_n + \tfrac{1}{2}c_2)$$
$$c_4 = h f(t_n + h, X_n + c_3)$$

This uses $t_n + \frac{1}{2}h$ as an intermediate point between t_n and $t_n + h$ (halfway) to develop successive approximations to $f(t, x)$, in contrast to Euler's method which just uses f at t_n, x_n to make the prediction for t_{n+1}, x_{n+1}.

Start at $x(0) = x_0 = 2$, $h = 0.2$, $f(t,x) = \frac{1}{x+t}$.

We have

$$c_1 = 0.2 \cdot \left(\frac{1}{2+0}\right) = 0.1$$

$$c_2 = 0.2\left(\frac{1}{0 + \frac{h}{2} + 2 + \frac{0.1}{2}}\right) = 0.2\left(\frac{1}{2.15}\right) = 0.093$$

$$c_3 = 0.2\left(\frac{1}{0.1 + 2 + \frac{0.093}{2}}\right) = 0.093175$$

$$c_4 = 0.2\left(\frac{1}{0.2 + 2 + 0.093175}\right) = 0.0872157$$

$$X_{n+1} = 2 + \frac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4) = 2.09326 \ldots \approx x(0.2).$$

To approx $x(0.4)$ and $x(0.6)$, apply the same method again i.e.

$$x_1 = 2.09326$$

$$c_1 = hf(t_1, x_1) = 0.2 \frac{1}{0.2 + 2.09326}$$

$$c_2 = hf(t_1 + \tfrac{h}{2}, x_1 + \tfrac{1}{2}c_1)$$

$$= 0.2 \frac{1}{0.3 + x_1 + \frac{1}{2}c_1}$$

and so on

$$x_2 = x_1 + \tfrac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4). \approx x(0.4)$$

Similarly for $x_3 \approx x(0.6)$, find c_1, c_2, c_3, c_4 starting from (t_2, x_2) to get x_3.

11. Systems of equations are not officially in this unit but we can extend Euler's method quite easily.

We have $\frac{dx}{dt} = xy + t$, $\frac{dy}{dt} = x - t$.

$$x(0) = 0, \quad y(0) = 1.$$

Euler's method for one uncoupled DE ($\dot{x} = f(t, x)$) uses f at t_n, x_n to predict x at t_{n+1}.

Here we do the same.

In general, say

$$\frac{dx}{dt} = f(x, y, t)$$

$$\frac{dy}{dt} = g(x, y, t)$$

then we can approximate

$$x_{n+1} = x_n + h f(x_n, y_n, t_n)$$

$$y_{n+1} = y_n + h g(x_n, y_n, t_n)$$

In this case, we have

$$x_0 = 0, \quad y_0 = 1, \quad h = 0.1$$

$$x_1 = 0 + (0.1) f(x=0, y=1, t=0)$$

$$= 0.1 \left[(0)(1) + 0\right] = 0$$

$$y_1 = 1 + 0.1(0-0) = 1.$$

$$x_2 = 0 + 0.1(x_1 y_1 + t_1) = 0 + 0.1(+0.1) = +0.01$$

$$y_2 = 1 + 0.1(x_1 - t_1) = 1 + 0.1(0 - 0.1) = 1 - 0.01 = 0.99$$

$$x_3 = x_2 + h f(x_2, y_2, t_2)$$

$$= 0.01 + 0.1(x_2 y_2 + t_2) = 0.01 + 0.1((0.01)(0.99) + 0.2)$$

$$= 0.03099$$

$$y_3 = y_2 + 0.1(x_2 - t_2) = 0.99 + 0.1(0.01 - 0.2) = 0.971$$

and so on to x_5, y_5.

12. Beyond the scope of this unit.