Sigma-Delta Modulators

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1 Description of Sigma-Delta modulators

Sigma-Delta ($\Sigma\Delta$) analog to digital converters are among the key components in modern electronics [1, 2]. Their main purpose is to provide cheap conversion from analog to digital signals. The main part of a $\Sigma\Delta$ converter is the $\Sigma\Delta$ modulator, which is represented in Figure 1. The discrete-time analog input r(k) has values in the interval [-1,1],the quantized output d(k) takes only two values, -1 and 1; f(k) is the approximation error, and D(z) is a causal dynamic system (usually chosen as a discrete time integrator), and its order gives the order of the $\Sigma\Delta$ modulator. The state space representation of this system is:

$$\begin{cases} x(k+1) = Ax(k) + bf(k) \\ s(k) = c^T x(k) \\ d(k) = sgn(s(k)) \\ f(k) = r(k) - d(k) \end{cases} \tag{1}$$

In the sequel we will show that the definition of the sgn function is a key point for the representation of (1) as Linear Complementarity (LC) system.

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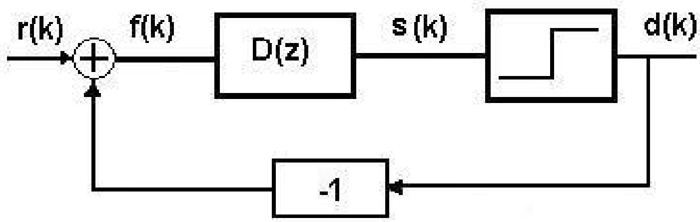

Figure 1: $\Sigma\Delta$ modulator scheme.

2 LC modelling

To represent system (1) as a LC system, we have to introduce some complementarity variables, that can help us to show the dynamic behavior of the modulator [3, 4]. Different definition of the sgn function correspond to different representations of $\Sigma\Delta$ as a LC system. If we assume the sgn definition reported in Figure 2, a representation of the modulator can be obtained by using four couples of complementarity variables:

$$\begin{cases} x(k+1) = Ax(k) + bf(k) \\ s(k) = c^T x(k) \\ f(k) = r(k) - d(k) \\ d(k) = \frac{1}{2}(u_2(k) - u_1(k)) \\ y_1(k) - y_2(k) = s(k) \\ y_3(k) = u_1(k) + u_2(k) - 2, \qquad u_3(k) = 1 \\ y_4(k) = u_1(k), \qquad u_4(k) = u_2(k) \\ 0 \leq u_i(k) \perp y_i(k) \geq 0 \qquad i = 1, ..4. \end{cases} \tag{2}$$

With this representation there are four possible cases, so as shown in Table 1. To understand the behavior of this representation we can consider that, since $u_3 = 1$, $y_3 = 0$ always, so we have $u_1 + u_2 = 2$, and the couple (u_4, y_4) is used to impose $u_1 \perp u_2$, so d can be only 1 or -1 . If $s > 0$, then $y_1 > y_2 \geq 0$, which implies $u_1 = 0$ and $u_2 = 2$, thus d=1. If $s = 0$, then $y_1 = y_2 = 0$, which implies $u_1 = 0$ and $u_2 = 2$, or $u_2 = 0$ and $u_1 = 2$, thus d can be respectively 1 or -1. If $s < 0$, then $y_2 > y_1 \geq 0$, which implies $u_2 = 0$ and $u_1 = 2$, thus d=-1.

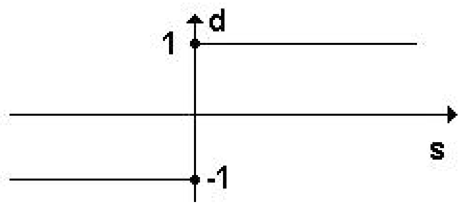

Figure 2: Definition of the sgn function.

s	d	u_1	u_2	y_1	y_2
>0	1	0	>0	>0	0
<0	-1	>0	0	0	>0
0	1	0	>0	0	0
0	-1	>0	0	0	0

Table 1: different input-output cases with the LC representation of the $\Sigma\Delta$ Modulator.

As explained above, this model has an ambiguity for s=0, where d can be 1 or -1, and this ambiguity is a problem if we want to use the LC model in a realistic application. In order to overcome this problem, we have to guarantee that the sgn function in s=0 assumes only one value, and express this with LC variables.

3 Open problem

If we decide that for s=0 we want d=1, from the previous LC model and Table 1 we can observe that this condition is expressed by the relation $u_2 + y_2 > 0$ (in fact the fourth line of the table doesn't satisfy this condition, and so for s=0, the value d=1 is the only one). This relation can't be used into the LC model because it isn't in the form of complementarity variables. So we can think to introduce another couple of complementarity variables, (u_5, y_5), to obtain an approximation of this form:

$$y_5(k) = u_2(k) + y_2(k) - \varepsilon, u_5(k) = 0, 0 < \varepsilon < 1.$$

The constant ε is the lowest representable positive number near 0. With this couple, $y_5 \geq 0$ always, so $u_2 + y_2 \geq \varepsilon > 0$, and the sgn function obtained is shown in the Figure 3.

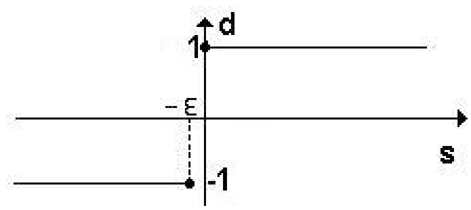

Figure 3: New definition of the sgn function.

The LC model for system (1) becomes:

$$\begin{cases} x(k+1) = Ax(k) + bf(k) \\ s(k) = c^T x(k) \\ f(k) = r(k) - d(k) \\ d(k) = \frac{1}{2}(u_2(k) - u_1(k)) \\ y_1(k) - y_2(k) = s(k) \\ y_3(k) = u_1(k) + u_2(k) - 2, \qquad u_3(k) = 1 \\ y_4(k) = u_1(k), \qquad u_4(k) = u_2(k) \\ y_5(k) = u_2(k) + y_2(k) - \varepsilon, \qquad u_5(k) = 0, \quad 0 < \varepsilon < 1 \\ 0 \leq u_i(k) \perp y_i(k) \geq 0 \qquad i = 1, ...5. \end{cases} \tag{3}$$

s	d	u_1	u_2	y_1	y_2
≥ 0	1	0	> 0	≥ 0	0
$\leq -\varepsilon$	-1	> 0	0	0	$\geq \varepsilon$

Table 2: different input-output cases with a second LC representation of the $\Sigma\Delta$ Modulator.

Now there are two cases, as shown in Table 2. A problem of this representation is that the sgn function isn't defined in the interval $(-\varepsilon, 0)$; thus the LC model found can be used in applications, by considering ε equal to the lowest representable number for a computer, and theoretically, considering that for each value of s, always exist a value for ε so that $s < -\varepsilon$ and so exist a corresponding value for d.

References

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