The buck converter as a complementarity problem

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Abstract

This is a preliminary report for the Fifth Framework research project IST-2002-422, "Simulation and Control of Nonsmooth Systems".

We formulate the step-down (*buck*) power converter in the framework of complementarity theory and show that it does not satisfy the conditions to ensure a unique solution.

1 Introduction

The basic step-down dc-dc power converter, also known as the *buck* converter, is displayed in Figure 1.

We assume an ideal diode, *i.e.* $i_d = 0$ if $v_d < 0$ and $v_d = 0$ if $i_d > 0$. The transfer of energy from the voltage source V_{in} to the load R is modulated by opening ($\bar{u} = 0$) or closing ($\bar{u} = 1$) the switch.

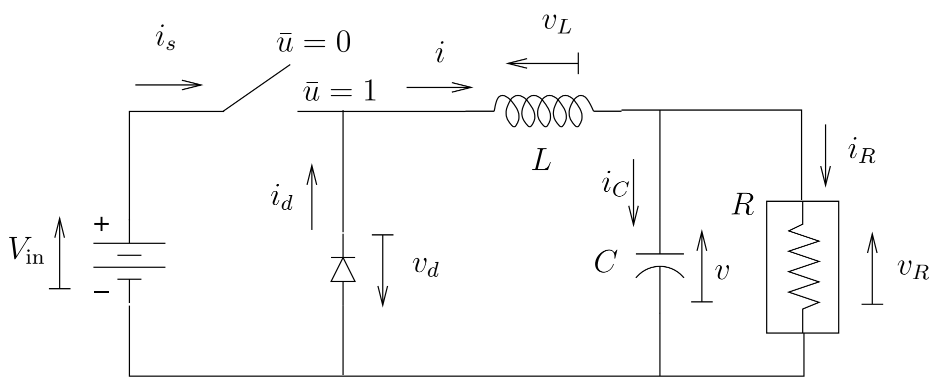

Figure 1: Basic scheme of the buck converter

Due to the ideal modelling of the diode and assuming $V_{\rm in} > 0$ one gets $i_d = 0$ and $v_d = -V_{\rm in} < 0$ when $\bar{u} = 1$ and $v_d = 0$ and $i_d > 0$ when $\bar{u} = 0$. Using i and v as state variables, Kirchoff's laws yield

$$\begin{aligned} L\frac{\mathrm{d}i}{\mathrm{d}t} + v &= \bar{u}V_{\rm in}, \\ C\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{R} &= i. \end{aligned} \tag{1}$$

Working with adimensional time τ, with $t = \tau\sqrt{LC}$, and with "current-like" state variables $x_1 = \sqrt{C/L}v$, $x_2 = i$ this can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \begin{pmatrix} -\gamma & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ E \end{pmatrix} \bar{u} \equiv Ax + b\bar{u}, \tag{2}$$

where

$$\gamma = \frac{1}{R}\sqrt{\frac{L}{C}}$$

is adimensional and $E = \sqrt{C/L}V_{\rm in}$ has dimensions of current.

The state of the switch ($\bar{u}$) depends on the value of a general function of the state variables and time

$$\bar{y} = f(x, \tau).$$

The simplest relation between $\bar{y}$ and $\bar{u}$ is

$$\bar{u} = \begin{cases} 1 & \text{if } \bar{y} < 0 \\ 0 & \text{if } \bar{y} > 0. \end{cases} \tag{3}$$

For a PWM voltage-controlled buck converter,

$$f(x, \tau) = x_1 - r(\tau),$$

with r a periodic function, while for a sliding-mode controlled buck

$$f(x, \tau) = x_2 + ax_1 - s(\tau)$$

with $s(\tau)$ constant for a regulation problem.

2 The buck converter as an LCS

Following [2] we introduce two pairs of complementarity variables [3] (w_1, z_1), (w_2, z_2) by means of

$$\bar{y} = w_2 - z_1, \tag{4}$$

$$\bar{u} = z_2, \tag{5}$$

$$z_1 = w_2 - f(x, \tau), \tag{6}$$

$$z_2 = 1 - w_1. \tag{7}$$

The complementarity condition $w \geq 0$, $z \geq 0$, $w \cdot z = 0$ has the following solutions:

($w_1 = w_2 = 0$) This implies $z_2 = 1$, $z_1 = -\bar{y} \geq 0$ and $\bar{u} = z_2 = 1$. Hence the dynamics is well defined in this case, with $\bar{u} = 1$.

($w_1 = z_2 = 0$) One immediately gets $0 = 1$ and hence there is no solution in this case.

($z_1 = w_2 = 0$) One gets $\bar{y} = -z_1 = 0$, $z_2 = \bar{u} \in [0, 1]$ and $w_1 = 1 - z_2 \in [0, 1]$. The dynamics is not defined since we do not get an unique value for $\bar{u}$.

($z_1 = z_2 = 0$) Now $\bar{u} = 0$, $w_1 = 1$ and $w_2 = \bar{y} \geq 0$, so again the dynamics is well defined, this time with $\bar{u} = 0$.

We conclude that the LCS formulation does not solve the problem of the lack of definition of the dynamics of the buck converter for a state such that $f(x, \tau) = 0$, *i.e.* a state on the boundary of the two linear dynamics.

3 The initial value problem for the buck converter as an LCP

The lack of wellposedness for the buck converter as a complementarity system can be also seen by considering the initial value problem, which yields an LCP [3].

Consider initial conditions (τ_0, x_0). At τ_0 the relation between u and y is

$$z_1 = w_2 - a, \tag{8}$$

$$z_2 = 1 - w_1, \tag{9}$$

where $a = f(x_0, \tau_0)$. This can be stated in the language of LCP theory

$$y = Mu + q$$

with

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} -a \\ 1 \end{pmatrix}.$$

It can be shown that if M is a $P-$matrix (all principal minors strictly positive) then the LCP has a unique solution and hence a unique initial $\bar{u}$ could in this case be recovered. Obviously, our M is not a $P-$matrix and hence the existence and uniqueness issue remains in principle open. However, we are going to show explicitly that uniqueness does not hold. One has several possibilities:

- $a < 0$. In this case $q > 0$ and it is well known that the LCP has then at least the trivial solution $u = 0$: $w_1 = w_2 = 0$, $z_1 = -a > 0$, $z_2 = 1 > 0$ and $\bar{u} = 1$.
- $a > 0$. We have the solution $w_1 = 1 > 0$, $w_2 = a > 0$, $z_2 = 0$, $z_1 = 0$ and $\bar{u} = 0$.
- $a = 0$. We can write two different solutions for this case:

  1. $w_1 = 0$, $z_2 = 1$, $w_2 = 0$, $z_1 = 0$ and $\bar{u} = 1$.
  2. $w_1 = 1$, $z_1 = 0$, $z_2 = 0$, $w_2 = 0$ and $\bar{u} = 0$.

  Hence, we cannot start the dynamical system if $a = 0$, *i.e.* if we have an initial condition on the boundary between dynamics.

4 Discussion

From the point of view of the LCP analysis of the initial condition, the problem arises from the zero $(1,1)-$minor of M. To solve this, z_1 has to depend on w_1 (this is what is considered in [2]). However, if we write

$$z_1 = \alpha w_1 + w_2 - f(x, \tau)$$

for some $\alpha > 0$, we immediately get

$$\bar{y} = f(x, \tau) + \alpha(\bar{u} - 1),$$

which does not make any sense with (3). Any way out of this seems to require an enlargement of the dynamics so that the vector field in the enlarged state space is well defined everywhere. Modelling the switch using the ideas of [1] can probably yield some hints.

References

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